

POLAR CYLINDRICAL AND POLAR SPHERICAL COMPLEX NUMBERS

SECTION 1 - Introduction

This document outlines and develops theory that makes it possible to utilize the same theory as k -cross (as the singular complex number) and the k -hat basis exponential (as the complex exponential) in any polar R^3 coordinate system. This documents asserts that the equivalent complex numbers in the polar cylindrical system are r -cross ($r\times$), θ -cross ($\theta\times$), and z -cross ($z\times$) whilst those in the polar spherical system are r -cross ($r\times$), θ -cross ($\theta\times$), and ϕ -cross ($\phi\times$). This mathematics means that it will be possible to appropriately choose the correct complex number for any situation, that is, if working in derived polars rather than the standard rectangular basis. This [possibly] will prove to be invaluable to electromagnetism.

There is an infinity of complex numbers, any one system defined by a **three-component-cross-product-triangle** and any one system transformed to any other by a change of basis, by what I am terming a **cisk-scaling** (**expk=cisk**). The complex numbers of the resulting system I assert are the resulting basis components of that system appended with cross $\times$, exactly as for k -cross, that is, the components of the **cross product triangle** that **defines the basis system**. The relevant item of theory for competent application in choosing the appropriate complex number [given that there is an infinity of them] is the accurate choice of system for a particular circumstance. This means an accurate specification of the coordinates of the plane of rotation and consequent specification of the orthogonal complement to the plane, the rotation axis. The basis vector of the rotation axis appended with $\times$ is then the complex number for the algebra of that plane. The most common being the standard system of $k\times$ on the x - y plane and the associated complex numbers $j\times$ and $i\times$ for the rectangular quaternion system, whilst this document seeks [I think successfully] to generalize this to arbitrary systems with an emphasis on the polar system of cylindrical bases $i\times$, $j\times$, $k\times \rightarrow r\times$, $\theta\times$, $z\times$. This may look like a change of base, and that is because that is exactly what it is which is interesting.

In polar cylindrical coordinates, the complex number for the plane of rotation of the radius (z -cross, $z\times$) is exactly the same as k -cross, $k\times$. This is why a Smith chart for impedance calculations is defined for the complex plane (since impedance Y is a complex number) but all of the calculations can be done in polars without invoking complex theory. This is because it is the same theory, there is no such thing as complex numbers. Again, what was thought of as the complex number i is the single component cross product $k\times$ which is a basis operator, not a scalar [since it is definitely associated with direction as well as magnitude]. Now that it is understood with absolute certainty that there is no extended field of complex numbers over the reals, I am now again utilizing the terminology of *complex number* since the concept has meaning with mathematicians in that it is a base for sines and cosines (via basis exponentials as expressed by Euler's formula). $k\times$ is the complex number for the x - y plane in rectangulars, whilst z -cross is the complex number for the r - θ plane in polars, and they are exactly the same thing as is demonstrated and proved a number of times throughout this document. The sines and cosines that are utilized in the r - θ plane in polars are the harmonics of the $k\times$ basis exponential [in which $k\times$ is implicit]. Hence the Smith chart is a chart of $k\times$ basis exponentials (k -hat-cross-product basis exponentials, **expk**).

I start out with this document by outlining that the polar cylindrical and spherical systems can be obtained by scaling the standard rectangular system of i -hat, j -hat, and k -hat by basis exponentials (**expi**, **expj**, and **expk**): a single basis exponential for the cylindrical system and two consecutive basis exponentials for the spherical system. This is essentially a change of basis [as per the *change of principal axis of rotation* document that I wrote several months back]. I then work essentially the same mathematics as for k -cross ($k\times$) and the k -hat basis exponential (**expk**) for the equivalent complex-number-like quantities in the derived polar systems, that of r -hat, θ -hat, and z -hat bases appended with $\times$ for the rotation operators (complex numbers) $r\times$, $\theta\times$, and $z\times$ and their basis exponentials **expr**, **expq**, and **expz** [polar rotation matrices].

Cylindrical coordinate systems: cisk-scaling the scleronomic rectangular basis

There are six ways to parameterise a cylindrical system, three for the clockwise (**cw**) cylindrical systems and three for counter-clockwise (**ccw**) cylindrical systems. For the counter-clockwise (**ccw**) systems, which are standard, the polars can be x - y plane (z -axis cylindrical), z - x plane (y -axis cylindrical), or y - z plane (x -axis cylindrical) with the cylindrical axis being the remaining mutually orthogonal axis to the plane [similarly as the latent rotation axis for $i = k\times$ in x - y plane scalar algebra]. This is the latent axis in the context of the complex plane where algebra [of R^2] had required the extension into R^3 for algebraic solutions [due to a negative determinant]. These six different cylindrical systems are parameterised by scaling the rectangular system by one of the three basis exponentials, **cisk** (**expk** = $e^{\theta\hat{k}\times}$) for the z -axis **ccw** cylindrical, **cisj** (**expj** = $e^{\theta\hat{j}\times}$) for the y -axis **ccw** cylindrical, or **cisi** (**expi** = $e^{\theta\hat{i}\times}$) for the x -axis **ccw** cylindrical [for x - y , y - z , or z - x], and the basis exponential conjugates for the **cw** systems. The cylindrical axis is the rotation axis and the pole of the polar system, that is, the basis vector [basis-hat] to which the basis exponential (**cisk**, **cisj**, or **ciski**) is defined and the rotation eigenvector around which the basis functions circulate, clockwise for scaling with **cisk**, **cisj**, or **ciski** and counter-clockwise for scaling with the conjugates **cisk'**, **cisj'**, or **ciski'**.

This will primarily concern parameterizing a **ccw** cylindrical system for the x - y plane with z being the cylindrical axis. This is **cisk-scaling** (and hence **expk**-rotating) the standard rectangular system, that is, scaling the rectangular system by the k -hat basis exponential **expk**. This is performed for the other cylindrical axes with the j -hat **expj** or i -hat exponential **expi** for which the theory is equivalent for those cylindrical systems.

Ordinarily, from the standard rectangular system $\vec{r} = (x, y, z)' = \hat{i}x + \hat{j}y + \hat{k}z$, to make the transformation from the rectangular system to cylindrical with **ccw polars** around the z-cylindrical axis, one would define $x = r \cos \theta$, $y = r \sin \theta$, $z = z$. The position vector is then

$$\begin{aligned} \vec{r} &= \hat{i}x + \hat{j}y + \hat{k}z \\ &= \hat{i}r \cos \theta + \hat{j}r \sin \theta + \hat{k}z \\ &= (r \cos \theta, r \sin \theta, z)' \end{aligned} \quad \text{i.e.,} \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix} = r \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{r}r + \hat{z}z$$

where $\hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta = (\cos \theta, \sin \theta, 1)'$ and $\hat{z} = \hat{k} = (0, 0, 1)'$. This hence gives two of the three cylindrical basis vectors ($\hat{r}$ -hat and $\hat{z}$ -hat) straight out as an identity for the position vector of the standard rectangular system expressed in polars. The third basis vector ($\hat{\theta}$ -hat) is obtained by crossing the two $\hat{\theta} = \hat{z} \times \hat{r}$ that is,

$$\hat{\theta} = \hat{z} \times \hat{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} \quad \text{so that} \quad \vec{r} = r \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{r}r + \hat{\theta}0 + \hat{z}z$$

Alternatively, to change the parameterisation to the cylindrical system, one could write

$$\begin{aligned} \vec{r} &= (x, y, z)' \\ &= \hat{i}x \left(\sqrt{x^2 + y^2} / \sqrt{x^2 + y^2} \right) + \hat{j}y \left(\sqrt{x^2 + y^2} / \sqrt{x^2 + y^2} \right) + \hat{k}z \\ &= \hat{i} \sqrt{x^2 + y^2} \left(x / \sqrt{x^2 + y^2} \right) + \hat{j} \sqrt{x^2 + y^2} \left(y / \sqrt{x^2 + y^2} \right) + \hat{k}z \\ &= \hat{i}r(x, y) \cos \theta(x, y) + \hat{j}r(x, y) \sin \theta(x, y) + \hat{k}z \end{aligned} \quad \begin{aligned} &\text{where} \quad r(x, y) = \sqrt{x^2 + y^2} \\ &\quad \cos \theta = x / \sqrt{x^2 + y^2} \\ &\quad \sin \theta = y / \sqrt{x^2 + y^2} \\ &\quad \theta(x, y) = \arctan(y / x) \end{aligned}$$

Note that this is an **identity**, that is $A \equiv A$ for each coordinate which may cause problems, for example, subtracting A from both sides leaves a null equation. This is overcome [in other contexts] by ensuring that the **spatial variable** is always related to a **basis variable**. The difference between an identity and an equation is the concept of relevance here. When this transformation is made, I have always thought of it as an equation but here, the mathematics from rectangulars to polars is definitely an identity which could lead to a contradiction. This is an identity because it is simply an algebraic rearrangement. It has taken one quantity, the position vector, and rearranged it into another form, and so relevant item is the difference between a **reparameterization** and a **transformation**.

The better way is where the basis is **cisk**-scaled and expressed as forward and backward variables where the initial coordinates are the backward spatial variables and the **cisk** scaling provides for an independent parameter for the change of basis, independent of the position vector [this in other contexts provides for a parameter that parameterizes a forward rotational transition, for example, the forward transition to the forward spatial variables]. The **cisk**-scaling is

$$\begin{aligned} e^{\theta \hat{k} \times} \vec{r}_0 &= e^{\theta \hat{k} \times} (x_0, y_0, z_0)' \\ &= e^{\theta \hat{k} \times} (\hat{i}x_0 + \hat{j}y_0 + \hat{k}z_0) \\ &= (e^{\theta \hat{k} \times} \hat{i})x_0 + (e^{\theta \hat{k} \times} \hat{j})y_0 + (e^{\theta \hat{k} \times} \hat{k})z_0 \\ &= (e^{\theta \hat{k} \times} \hat{i})x_0 + (e^{\theta \hat{k} \times} \hat{j})y_0 + \hat{k}z_0 \\ &= (\hat{i} \cos \theta + \hat{j} \sin \theta)x_0 + (-\hat{i} \sin \theta + \hat{j} \cos \theta)y_0 + \hat{k}z_0 \\ &= \hat{i}(x_0 \cos \theta - y_0 \sin \theta) + \hat{j}(x_0 \sin \theta + y_0 \cos \theta) + \hat{k}z_0 \\ &= \hat{i} \sqrt{x_0^2 + y_0^2} \cos(\theta + \arctan y_0 / x_0) + \hat{j} \sqrt{x_0^2 + y_0^2} \sin(\theta + \arctan y_0 / x_0) + \hat{k}z_0 \quad \text{phasor form} \\ &= \hat{i}r_0 \cos(\theta + \phi_0) + \hat{j}r_0 \sin(\theta + \phi_0) + \hat{k}z_0 \quad r_0 = \sqrt{x_0^2 + y_0^2} \quad \text{and} \quad \phi_0 = \arctan(y_0 / x_0) \\ &= r_0 (\hat{i} \cos(\theta + \phi_0) + \hat{j} \sin(\theta + \phi_0)) + \hat{k}z_0 \\ &= \hat{r}r_0 + \hat{z}z_0 \quad \hat{r} = \hat{i} \cos(\theta + \phi_0) + \hat{j} \sin(\theta + \phi_0) \quad \text{and} \quad \hat{z} = \hat{k} \end{aligned}$$

and so allows for backward variables expressed in radians and phase. The line from rectangular to phasor form, that is, rectangular (x, y) to magnitude-phase (r, θ) , proceeds like this

$$\begin{aligned} X &= x_0 \cos \theta - y_0 \sin \theta \\ &= r_0 \cos(\theta + \phi_0) \\ &= r_0 \cos \theta \cos \phi_0 - r_0 \sin \theta \sin \phi_0 \end{aligned} \quad \begin{aligned} &\text{or equivalently} \\ Y &= x_0 \sin \theta + y_0 \cos \theta \\ &= r_0 \sin(\theta + \phi_0) \\ &= r_0 \sin \theta \cos \phi_0 + r_0 \cos \theta \sin \phi_0 \end{aligned}$$

$$x_0 = r_0 \cos \phi_0 \quad y_0 = r_0 \sin \phi_0 \quad \text{hence similarly} \quad x_0 = r_0 \cos \phi_0 \quad y_0 = r_0 \sin \phi_0$$

$$\text{so that} \quad r_0^2 = r_0^2 \cos^2 \phi_0 + r_0^2 \sin^2 \phi_0 = x_0^2 + y_0^2 \quad \text{and} \quad y_0 / x_0 = \sin \phi_0 / \cos \phi_0 = \tan \phi_0$$

I am of the opinion that this is the better way to transform to polar cylindricals because (1) it is **consistent with cis-scaling**, (2) initial conditions of the position vector are expressed in terms of **magnitude and phase**, and (3) has θ as a **free parameter** for the cylindrical transformation, which can be differentiated independently [from the initial conditions] and where the initial angle is given by phase.

The **standard rectangular system** is **scleronomic** which here means that time dots (total time derivatives) for the components are null. The cylindrical and spherical coordinate systems are **derived systems**. These are assumed to be **rheonomic** bases, where time dots are not null, until stated otherwise. If θ of the transformation is constant rather than time dependent, the derived system is **derived scleronomic** whilst if θ is time dependent it is **derived rheonomic**. For **derived rheonomic** where the cis argument theta θ is time dependent, this implies a spinning (continuously rotating) system or oscillating (rotating back and forth) system and a continuous ongoing change of base. For **derived scleronomic** where theta θ is constant, this implies a single fixed angular change of orientation and the associated single change of base [from k -cross to j -cross for example, $k \times \rightarrow j \times$]. The terms **scleronomic** and **rheonomic** are from **Lagrangian theory** but I think that they are well suited to the **theory of basis systems** as discussed here. In **Lagrangian theory**, the terms **scleronomic** and **rheonomic** relate to **constraints** [whilst I am using the terms here for features of coordinate systems]. Basis systems essentially constrain a system to a certain space generated by the basis vectors. I therefore think I might end up getting a one-for-one correspondence with the Lagrangian theory. Stating that the rectangular system is scleronomic at the outset means that vectors and outer products of vectors in the [scleronomic] rectangular system can be differentiated component-wise without having to dot the basis terms also, and provides a concise way of saying so. All basis systems derived from the scleronomic standard rectangular system are rheonomic until stated otherwise, and hence for the derived rheonomic basis systems, the basis components are differentiated with the components (as product rules for example). For the velocity vector, differentiating the position vector

$$\dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{r} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + r \dot{\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} + \dot{z} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{r} \\ r \dot{\theta} \\ \dot{z} \end{pmatrix} \quad \hat{i} \dot{x} + \hat{j} \dot{y} + \hat{k} \dot{z} = \hat{r} \dot{r} + \hat{\theta} r \dot{\theta} + \hat{z} \dot{z}$$

and hence with cylindrical rheonomic basis $\hat{i} \dot{x} + \hat{j} \dot{y} + \hat{k} \dot{z} = \hat{r} \dot{r} + \hat{\theta} r \dot{\theta} + \hat{z} \dot{z}$ noting that r , θ , and z are scalar (so too then are r -dot, θ -dot, and z -dot) and the theta-hat component is there because it is assumed rheonomic. The cylindrical rheonomic basis vectors r -hat, θ -hat, and z -hat are then

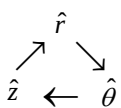
$$\hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} = \hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0 \quad \hat{\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} = -\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0 \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{i} 0 + \hat{j} 0 + \hat{k} 1$$

The distinction between rheonomic and scleronomic derived systems is not all that important to the rest of this document but prepares this theory for it to be used in what I hope to do with it in the context of vector calculus. I am writing all of this with the hope to turn it all into calculus where the distinction between **rheonomic and scleronomic derived systems**, and the ability to say so concisely, will be invaluable.... that is, whether a change of base to polars is scleronomic (fixed change of orientation) or rheonomic (spinning or oscillating).

The form of the cylindrical basis components, obtained by the several methods above consequently

demonstrate that the polar cylindrical basis components are the columns of the k -hat exponential as these are the vectors of the k -hat basis exponential **expk** defined for, respectively, the i -hat, j -hat, and k -hat directions. Hence, the k -hat exponential (complex exponential and **expk** = **cisk** scaling) applied to the scleronomic rectangular basis is a transformed coordinate system.

Crossing r -hat with θ -hat gives z -hat, crossing θ -hat with z -hat gives r -hat, and crossing z -hat with r -hat gives θ -hat. Hence, the basis components have the properties $\hat{r} \times \hat{\theta} = \hat{z}$, $\hat{\theta} \times \hat{z} = \hat{r}$, and $\hat{z} \times \hat{r} = \hat{\theta}$ and therefore form the **derived rheonomic right-handed cross product triangle**



$$\vec{r} = (x, y, z)' = \hat{i} x + \hat{j} y + \hat{k} z$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \hat{i} \dot{x} + \hat{j} \dot{y} + \hat{k} \dot{z}$$

that is, it is a **rheonomic derivative** of the **scleronomic rectangular system** (the **rheonomic system derived from the underlying right handed standard rectangular system**). Therefore, **circular scaling** the **scleronomic rectangular system** by the k -hat exponential, that is by **cisk**, converts the standard rectangular basis components of the x - y plane into the cylindrical polars basis components around the z axis of k -hat (the complex number for that plane). This suggests a new form of mathematics of doing exactly

that, scaling the cross-product triangle of a coordinate system by basis-hat exponentials to transform from that **cross product triangle** to the **cross-product triangle** of the **derived rheonomic system**. This is what I have called **cis-scaling** a basis. The **single component cross products** ($k\times$ for example) of the rectangular system are essentially what was thought of as standard complex numbers, similarly are the cylindrical cross products [and also similarly spherical]. Noteworthy is that these, for the generalized systems, are not single component [cross products] when expressed in standard rectangulars. Given that I demonstrated that a **quaternion** [in the standard basis] is actually a **three component [standard] cross-product**, polar cylindrical complex numbers [as cross products] have features of quaternions rather than complex numbers because the basis vectors of the derived system [$\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat], whilst unit vectors, are not single component (other than $\hat{z}$ -hat, when expressed in terms of the standard rectangular basis). The distinction between complex number and quaternion for the generalized system seemed therefore to become blurred for the generalized systems. For an updated definition of complex number and quaternion, these concepts can be defined by the relevant cross-product triangle: a single component of that cross product triangle [so it is single component within that system] is complex-number-like whilst if it is a combination of components then it is quaternion-like [but operators, not numbers or scalars]. I posit, therefore, that the quintessential feature of a complex number in a **generalized coordinate system** is that **squares return their negative** [which single component cross products do within the generalized basis system] and the quintessential feature of a **generalized basis system** is the **cross product triangle**.

From this discussion and previous parts of this document (**k-cross and the k-hat exponential**) the components of the derived basis have the properties that $(\hat{r}\times)^2(\cdot) = -1(\cdot)$ when applied to $\hat{\theta}$ -hat and $\hat{z}$ -hat, $(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$ when applied to $\hat{z}$ -hat and $\hat{r}$ -hat, and $(\hat{z}\times)^2(\cdot) = -1(\cdot)$ when applied to $\hat{r}$ -hat and $\hat{\theta}$ -hat. This last complex number ($\hat{z}$ -cross) is the same as k -cross since it is the cross product basis vector of the same rotation axis, here k -hat corresponds to that of the polar axis $\hat{z}$ -hat. This means that, when also considering sphericals, there are 9 types of complex number [3 for standard rectangular, 3 for polar cylindrical, and 3 for polar spherical], with one redundancy of that of k -cross and $\hat{z}$ -cross. For the cylindrical system in component form

$\hat{i} \times \hat{j} = \hat{k}$	$\hat{j} \times \hat{k} = \hat{i}$	$\hat{k} \times \hat{i} = \hat{j}$	scleronomic standard rectangular system
$e^{\theta\hat{k}\times}(\hat{i} \times \hat{j} = \hat{k})$	$e^{\theta\hat{k}\times}(\hat{j} \times \hat{k} = \hat{i})$	$e^{\theta\hat{k}\times}(\hat{k} \times \hat{i} = \hat{j})$	cisk -scaling the rectangular system
$(e^{\theta\hat{k}\times}\hat{i}) \times (e^{\theta\hat{k}\times}\hat{j}) = e^{\theta\hat{k}\times}\hat{k}$	$(e^{\theta\hat{k}\times}\hat{j}) \times (e^{\theta\hat{k}\times}\hat{k}) = e^{\theta\hat{k}\times}\hat{i}$	$(e^{\theta\hat{k}\times}\hat{k}) \times (e^{\theta\hat{k}\times}\hat{i}) = e^{\theta\hat{k}\times}\hat{j}$	distributing the expk = cisk
$\hat{r} \times \hat{\theta} = \hat{z}$	$\hat{\theta} \times \hat{z} = \hat{r}$	$\hat{z} \times \hat{r} = \hat{\theta}$	derived rheonomic cylindrical system

in other words

$$e^{\theta\hat{k}\times} \left(\begin{array}{ccc} & \hat{i} & \\ \nearrow & & \searrow \\ \hat{k} & \leftarrow & \hat{j} \end{array} \right) = \begin{array}{ccc} & e^{\theta\hat{k}\times}\hat{i} & \\ \nearrow & & \searrow \\ e^{\theta\hat{k}\times}\hat{k} & \leftarrow & e^{\theta\hat{k}\times}\hat{j} \end{array} = \begin{array}{ccc} & \hat{r} & \\ \nearrow & & \searrow \\ \hat{z} & \leftarrow & \hat{\theta} \end{array} \quad \text{that is,} \quad \begin{array}{ccc} & \hat{i} & \\ \nearrow & & \searrow \\ \hat{k} & \leftarrow & \hat{j} \end{array} \xrightarrow{\text{cisk}\theta} \begin{array}{ccc} & \hat{r} & \\ \nearrow & & \searrow \\ \hat{z} & \leftarrow & \hat{\theta} \end{array}$$

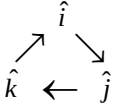
since $e^{\theta\hat{k}\times}\hat{i} = \hat{i} \cos \theta + \hat{j} \sin \theta = \hat{r}$, $e^{\theta\hat{k}\times}\hat{j} = \hat{j} \cos \theta - \hat{i} \sin \theta = \hat{\theta}$, and $e^{\theta\hat{k}\times}\hat{k} = \hat{k} = \hat{z}$, which [the “cis-kay-scaling”] has hence transformed the scleronomic rectangular basis [and hence complex numbers] to a derived rheonomic cylindrical basis [and set of complex numbers] with scaling by **cisk**, that is, by scaling [rotating] the rectangular system by the k -hat basis exponential. Note that the k -hat exponential has distributed over each of the rectangular basis-hats in the cross products of i -hat and j -hat, j -hat and k -hat, and k -hat and i -hat respectively, for example, $e^{\theta\hat{k}\times}(\hat{i} \times \hat{j} = \hat{k}) \Rightarrow (e^{\theta\hat{k}\times}\hat{i}) \times (e^{\theta\hat{k}\times}\hat{j}) = e^{\theta\hat{k}\times}\hat{k}$ and hence

$$e^{\theta\hat{k}\times} \left(\begin{array}{ccc} & \hat{i} & \\ \nearrow & & \searrow \\ \hat{k} & \leftarrow & \hat{j} \end{array} \right) = \begin{array}{ccc} & e^{\theta\hat{k}\times}\hat{i} & \\ \nearrow & & \searrow \\ e^{\theta\hat{k}\times}\hat{k} & \leftarrow & e^{\theta\hat{k}\times}\hat{j} \end{array}$$

Scaling the basis-cross product triangle by **expk**, at minimum, is a new way of saying so mathematically. The term “derived” in this context means a basis system derived from the standard rectangular system [I like the word derived from financial derivatives]. The most important results of this document can now be summarised as

rectangular complex numbers	cylindrical complex numbers	spherical complex numbers
$(\hat{k}\times)^2(\cdot) = -1(\cdot)$	$(\hat{r}\times)^2(\cdot) = -1(\cdot)$	$(\hat{r}\times)^2(\cdot) = -1(\cdot)$
$(\hat{j}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$
$(\hat{i}\times)^2(\cdot) = -1(\cdot)$	$(\hat{z}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\phi}\times)^2(\cdot) = -1(\cdot)$

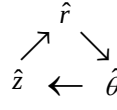
scleronomic
rectangular



rectangular complex
numbers

$\longrightarrow$
cisk θ

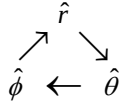
rheonomic
cylindrical



cylindrical complex
numbers

$\longrightarrow$
cisj ϕ

rheonomic
spherical



spherical complex
numbers

So that, in addition to the standard rectangular system,

for the **cylindrical system**

basis components

r-cross, θ -cross, and z-cross

sinr, sinq, sinz

cosr, cosq, cosz

tanr, tanq, tanz

secr, secq, secz

cisr, cisq, cisz

$\hat{r}$

$\hat{r} \times (\cdot) = \hat{r} \times$

sinr t

cosr t

tanr t

secr t

cisr $(\cdot) = \exp(\hat{r} \times)(\cdot)$

$\hat{\theta}$

$\hat{\theta} \times (\cdot) = \hat{\theta} \times$

sin θ t

cos θ t

tan θ t

sec θ t

cis θ $(\cdot) = \exp(\hat{\theta} \times)(\cdot)$

$\hat{z}$

$\hat{z} \times (\cdot) = \hat{z} \times$

sinz t

cosz t

tanz t

secz t

cisz $(\cdot) = \exp(\hat{z} \times)(\cdot)$

and for the **spherical system**

basis components

r-cross, θ -cross, and ϕ -cross

sinr, sinq, sinp

cosr, cosq, cosp

tanr, tanq, tanp

secr, secq, secp

cisr, cisq, cisp

$\hat{r}$

$\hat{r} \times (\cdot) = \hat{r} \times$

sinr t

cosr t

tanr t

secr t

cisr $(\cdot) = \exp(\hat{r} \times)(\cdot)$

$\hat{\theta}$

$\hat{\theta} \times (\cdot) = \hat{\theta} \times$

sin θ t

cos θ t

tan θ t

sec θ t

cis θ $(\cdot) = \exp(\hat{\theta} \times)(\cdot)$

$\hat{\phi}$

$\hat{\phi} \times (\cdot) = \hat{\phi} \times$

sin ϕ t

cos ϕ t

tan ϕ t

sec ϕ t

cis ϕ $(\cdot) = \exp(\hat{\phi} \times)(\cdot)$

CYLINDRICAL COMPLEX NUMBERS: the r -cross, θ -cross, and z -cross rotation operators

The **rotational operators** are more **quaternion-like** than complex-number-like when there is more than one of them operating, and so are operating on more than one orthogonal axes simultaneously [this means, unfortunately, that de-Moivre's theorem can't be invoked in such circumstances]. The **rotation operators** are **complex-number-like** when single component so that only one of the axes is being rotated, and so squares return their negative for the basis components other than its own direction, the **square taking the value of negative one** being the **quintessential attribute** to the workings of what was thought of as a complex number [so one could take the square root of a negative quadratic determinant]. This is also the condition for de-Moivre's theorem and a continuously rotating layer of a Rubiks cube.

The components of the derived rheonomic cylindrical r -hat, θ -hat, and z -hat basis are single component basis vectors within the basis system defined as $\hat{r} \times \hat{\theta} = \hat{z}$, $\hat{\theta} \times \hat{z} = \hat{r}$, and $\hat{z} \times \hat{r} = \hat{\theta}$ but not when expressed in terms of other bases including the standard basis (since the components are sinusoidally weighted linear combinations of the standard basis). So when do the components in an arbitrary derived basis system, appended with $\times$, have properties of that can be considered complex-number like and when do they have properties that can be considered quaternion-like?

The difference between a complex number and a quaternion in different coordinate basis systems where the basis vectors are not single component [when expressed in standard rectangulars] is determined by the cross-product triangle, and so is determined as such in their own basis system. They are single-component in that system, so a complex-number is the single component cross-product in that system whilst a quaternion is a combination of more than one of the component cross products in that system.

The square returning the negative has been the quintessential attribute of what was thought of by mathematicians as the algebraic complex number i for almost 500 years and this property holds good, in both the theory of k -cross for the standard rectangular system and that of r -cross in derived polar systems. Despite polar bases not being single component [when expressed in terms of i -cross, j -cross, and/or k -cross], all of the theory for polars works the same, amazingly, because of the properties that make the basis-crosses complex-number-like, that is, the cross-product-triangle and consequently single component within that system with the resulting quintessential feature [required for negative quadratic determinants] that

$$(\hat{r} \times)^2(\cdot) = -1(\cdot), \quad (\hat{\theta} \times)^2(\cdot) = -1(\cdot), \quad \text{and} \quad (\hat{z} \times)^2(\cdot) = -1(\cdot)$$

This quintessential property holds good for the derived system, so long as the components are expressed in terms of r -hat, θ -hat, and z -hat and, when appended with $\times$ for the cross product triangle, operate on other r -hat, θ -hat, and z -hat so that $(\hat{r} \times)\hat{\theta} = \hat{z}$, $(\hat{\theta} \times)\hat{z} = \hat{r}$, and $(\hat{z} \times)\hat{r} = \hat{\theta}$. This is the required conditions for the complex-number-like properties to hold good within derived systems. With this

condition met, that the cross-products remain within the basis system's own **cross products triangle**, means that the theory all works for any coordinate system derived from the standard basis, in the same way that it does for the standard basis itself. This is therefore the reason why the notion of **generalized coordinate system** can be invoked to solve mathematical problems. This then allows mathematicians to do **complex number and quaternion mathematics** in such systems, that is, in any **generalized coordinate system** so long as it has a **cross-product triangle**. Simple yet powerful idea. Hence, since $(\hat{r}\times)^2(\cdot) = -1(\cdot)$ (for θ -hat and z -hat), $(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$ (for z -hat and r -hat), and $(\hat{z}\times)^2(\cdot) = -1(\cdot)$ (for r -hat and θ -hat). Therefore, r -cross, θ -cross, and z -cross are the complex numbers when working in cylindrical polars. Further, for any **generalized basis system** defined by a **cross-product triangle**, the **complex numbers** for that system are the components of that system appended with $\times$. Again, I think it is important to stress the fact that these are operators, not scalars as was thought for the complex number $i = k\times$. This quantity IS NOT A NUMBER NOR A SCALAR, it only appears to be, a theorem for which I propose at the end of this document. The complex number $i = k\times$ IS AN OPERATOR, and to say otherwise is wrong: it operates implicitly on the x and y axes and with the latent z axis when R^2 is assumed...it is definitely not a number since it is an operator and it is definitely not a scalar since it is associated with direction. The theorem at the end of this document states the conditions under which $i = k\times$ (and equivalent in derived systems) can be treated mathematically as a number and algebraic scalar despite the fact that it is not.

- (1) The operator and complex number for the polar plane z -cross, $z\times$, corresponds to the rotation (pole) axis of the cylindrical system, that is, the complex number for the cylindrical axis [for rotations around that axis on a polar defined plane].
- (2) The operator r -cross corresponds to the spin axis of the cylindrical radius, that is, the complex number for a plane that is cut mutually perpendicularly to the cylindrical radius.
- (3) The operator θ -cross is less intuitive but, since the coordinates are mutually orthogonal, can be explained as the complex number to whatever the axis is that is at right angles to both the cylindrical axis and cylindrical polar radius at whatever their mutual orientation is. This means that θ -cross is essentially the complex number for an axis that is 90 degrees out of phase with the cylindrical polar radius on the plane that cuts the cylinder laterally [that which the polar radius is on]. These are essentially the x and y axes that have been rotated since y and x in terms of a circle on the x - y plane are out of phase by 90 degrees by definition [note that because the axes are orthogonal, it remains rectangular].

$z\times$ is the complex number if a negative determinant turns up solving algebraic quantities in r - θ coordinates and hence the complex number that is required for (what appears to be) scalar mathematics in these coordinates. $r\times$, $\theta\times$, and $z\times$ are also the relevant attributes that give the **basis-hat sines and cosines**, as well as the **basis-hat exponentials**, their rotational properties in polar cylindrical coordinates [rotated or rotating standard basis]. In an equation where a determinant is found to be negative when working in one of the planes orthogonal to r - θ polar plane [the less common r - z or θ - z planes], r -cross corresponds to rotations about the polar radius (generated by r -hat orthogonal to the θ - z plane) and θ -cross corresponds to rotations about the cylindrical axis (generated by θ -hat orthogonal to the r - z plane), that which rotates around the second orthogonal axis of the polar plane [orthogonal to r -hat].

An r -hat exponential could be the definition of a circularly polarized wave from such a source and the double-cross ($r \times r \times$) being the return echo of a radar pulse. I'll define the r -hat, θ -hat, and z -hat sinusoids now, **cosr**, **cosq**, **cosz**, **sinr**, **sinq**, and **sinz**, which are defined similarly as for k -hat, j -hat, and i -hat for the standard rectangular system. I then define the r -hat, θ -hat, and z -hat exponentials **expr**, **expq**, and **expz**, the exponentials of r -cross, θ -cross, and z -cross. These are the polar cylindrical rotation matrices which I demonstrate by diadic expansion.

CYLINDRICAL HARMONICS: the r -hat, θ -hat, and z -hat sines and cosines

Because the derived cylindrical systems form coordinate systems that have a cross product triangle, consisting of **r -cross**, **θ -cross**, and **z -cross** (**ϕ -cross** for the spherical system), and are hence complex numbers as to how complex numbers were understood as solutions to quadratics and the further defined by Euler's formula, this then means that there are further sinusoidal basis operators, that is, more forms of basis-hat sine and cosine. For sines these are **sinr**, **sin θ** , and **sinz** (this one, **sinz**, is the same as **sink**), as well as **sin ϕ** for the spherical system. As a suggested pronunciation, these are spoken for example as "sine-arh", "sine-theta", "sine-zed", and "sin-phi" or alternatively as "sine-radius", "sine-axis", "sine-azimuth", and "sine-colatitude". Those for cosine are **cosr**, **cos θ** , and **cosz** (same as **cosk**) and **cos ϕ** , with the suggested pronunciation "cos-arh", "cos-theta", "cos-zed", and "cos-phi" or alternatively as "cosine-radius", "cosine-axis", "cosine-azimuth", and "cosine-colatitude". There are also the basis exponentials **cisr**, **cis θ** , **cisz**, and **cis ϕ** , ("sis-arh", "sis-que", "sis-zed", and "sis-eff" or "sis-radius", "sis-azimuth", "sis-axis", and "sis-colatitude") as well as the double crosses, nilpotent mixed double crosses, and quaternion mixed triple crosses for example. This may end up being important mathematics for applied working, for instance, almost all of weather dynamics is rotational systems in cylindrical and spherical coordinates where spherical coordinates are frequently simplified to cylindrical coordinates as for the geostrophic and gradient wind. Proving the non-derivative properties of these basis operators is practically the same as for the rectangular system since all of these properties are based solely on the **properties of the relevant cross-product triangle**. For this one of the six cylindrical rheonomic basis systems, $z\times$ (z -cross) corresponds one-for-one with (k -cross) $k\times$.

Note that scalar harmonics in the radius r as a function of theta θ are polar roses, and these in turn are generalized circles. From the theory of differential geometry or similar, a radial polar function $r(\theta)$ can be transformed back to rectangulars by multiplying the radius by cosine for the x -component and multiplying the radius by sine for the y -component for a frequency of 1 Hz. For example,

the polar equation of a cardioid (heart shaped curve, the simplest example of a non-trivial epicycloid) is $r = 1 + \cos\theta$. This is written in x-y standard rectangulars as $x = r \cos\theta = (1 + \cos\theta) \cos\theta$ and $y = r \sin\theta = (1 + \cos\theta) \sin\theta$. The cosine and sine can be given an arbitrary frequency (other than 1Hz) which generates some further very intricate patterns [of a generalized circle], beyond those of a cardioid when defined for a cardioid radius. Polar roses are also radiation patterns [for symmetric lobes], for example, radio-frequency from a dipole antenna [transmitter]. These patterns, including for radar, are polar plots of radius over scanned azimuth theta. The complex number for this plane [the latent axis basis component appended with $\times$] is z-cross ($z\times$), which is equivalent to k-cross ($k\times$) [the same complex number for both the x-y plane and the r- θ polar plane]. The polar k-cross $k\times$, that is z-cross $z\times$, would also be the complex number for the cardioid example as well [whilst in polars], if the cosine was to be expanded or written in terms of complex exponentials [the z-hat exponential and conjugate **expk** and **expk'** in this instance], since it is a plane polar radius as a function of theta. In this instance, the relevant basis functions would be

$$\text{cosz } \theta = 1 + \frac{1}{2!} \theta^2 (\hat{z}\times)^2 + \frac{1}{4!} \theta^4 (\hat{z}\times)^4 + \dots \quad \text{and} \quad \text{expz } \theta = 1 + \theta \hat{z}\times + \frac{1}{2!} \theta^2 (\hat{z}\times)^2 + \frac{1}{3!} \theta^3 (\hat{z}\times)^3 + \frac{1}{4!} \theta^4 (\hat{z}\times)^4 + \dots$$

$$= \text{cosz } \theta + (\text{sinz } \theta) \hat{z}\times$$

This idea can be extended to sphericals for further patterns of curves and surfaces, and for scans over elevation [for example home chain also deducing elevation in addition to bearing and range in the early days of radar].

Unlike for the rectangular functions for which theta (for azimuth or phi for colatitude) is most appropriate, it is unclear what the best arbitrary argument is for the rheonomic functions. These systems are almost always parameterized. So whatever it is, when the time comes to parameterise [as required for vector calculus in these systems], it usually ends up being either constant (scleronomic) or linear with time t (rheonomic) since harmonics are almost always expressed in units of cycles per second, Hertz. For this reason, these will be expressed as functions of t . Since theta is usually an argument, saying sine-theta can be misconstrued as an argument when here it is a basis vector (basis-hat) and a basis-cross-product for theta-cross. The basis harmonics and basis-hat exponential for theta-hat I have considered terming **sinq**, **cosq**, and **cisq** since q is the roman form of theta. These are [suggested] pronounced **sine-que**, **cos-que**, and **sis-que** respectively. Actually going to keep with θ for now [changed back, going to use q in the vector definition, i.e., with **sinq**, **cosq**, and **expq** rather than **sin θ** , **cos θ** , and **exp θ**], but someone may have a slightly different preference if they start working with this mathematics [with θ -cross and θ -hat sinusoids **sin θ** ($\cdot$) and exponentials **cis θ** ($\cdot$) for example - whatever works best... I am now of the opinion that **sine-azimuth**, **cos-azimuth**, and **sis-azimuth** is best, then pronounced with an argument for example **sine-azimuth-omega-tee** for **sinq**(ωt)].

All of the maths was worked for the scleronomic standard rectangular system (k-cross and the k-hat exponential) but by the virtue of all of this mathematics being a consequence of a **basis system** with a **cross product triangle**, all of the mathematics carries over almost one-for-one with the derived (cis-scaled) rheonomic systems **[rotated ($\theta = \text{const}$) or rotating ($\theta = \omega t$) rectangular systems]**. This essentially lends itself to the concept of **generalized coordinates**, that is, **generalized basis coordinate systems**. In other words, if a **basis system** has a **cross product triangle**, then all of the mathematics is the same, including all of the **complex-number-like** properties. This is similarly as **Lagrangian theory** for **equations of motion** in the different **coordinate systems** where all of the mathematics is the same one-for-one. For this reason I was going to omit all of the working but have instead decided to include it... A summary table of these basis functions and their components in their respective directions is below.

Figure 1 Summary: cylindrical r-hat, θ -hat, and z-hat sines and cosines

Scalar function	r-hat operators p ($r\times$)	θ -hat operators p ($\theta\times$)	z-hat operators p ($z\times$)
sin t	(sinr t) $\hat{r} = \hat{r} t$	(sinθ t) $\hat{r} = \hat{r} \sin t$	(sinz t) $\hat{r} = \hat{r} \sin t$
	(sinr t) $\hat{\theta} = \hat{\theta} \sin t$	(sinθ t) $\hat{\theta} = \hat{\theta} t$	(sinz t) $\hat{\theta} = \hat{\theta} \sin t$
	(sinr t) $\hat{z} = \hat{z} \sin t$	(sinθ t) $\hat{z} = \hat{z} \sin t$	(sinz t) $\hat{z} = \hat{z} t$
cos t	(cosr t) $\hat{r} = \hat{r}$	(cosθ t) $\hat{r} = \hat{r} \cos \theta$	(cosz t) $\hat{r} = \hat{r} \cos t$
	(cosr t) $\hat{\theta} = \hat{\theta} \cos \theta$	(cosθ t) $\hat{\theta} = \hat{\theta}$	(cosz t) $\hat{\theta} = \hat{\theta} \cos t$
	(cosr t) $\hat{z} = \hat{z} \cos \theta$	(cosθ t) $\hat{z} = \hat{z} \cos t$	(cosz t) $\hat{z} = \hat{z}$

Each of these are obtained by the power series expansions, as per James Gregory, but with the single component basis cross products (r-cross [$r\times$], θ -cross [$\theta\times$], and z-cross [$z\times$]) and using the complex number-like properties of the basis-crosses (for r-cross, θ -cross, and z-cross) as per in earlier documents for the standard rectangular basis system, the most relevant of these properties being the quintessential feature of complex i

$$(\hat{r}\times)^2(\cdot) = -1(\cdot), \quad (\hat{\theta}\times)^2(\cdot) = -1(\cdot), \quad \text{and} \quad (\hat{z}\times)^2(\cdot) = -1(\cdot)$$

I am probably going to omit working [as I was saying] because it is evident, as per a **generalized basis coordinated system** and the **common properties of all cross product triangles...** I have gone ahead and done it all and it is in the next couple of dozen pages after the discussion of the cylindrical basis exponentials. I then diadically expand these into polar matrix basis operators, and then expand into standard rectangulars, and then provide a means of doing the polar basis mathematics by linear algebra which, again, is further unification of complex analysis with linear algebra [the staple of second year university and engineering mathematics].

CYLINDRICAL EXPONENTIALS: the $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat basis exponentials (polar rotation matrices)

For the standard rectangular scleronomic basis system, there are three formulae for Euler's formula for what was understood as a standard complex number as discussed in k -cross and the k -hat exponential: Euler's formula for $k \times$ (**expk=cisk**), Euler's formula for $j \times$ (**expj=cisj**), and Euler's formula for $i \times$ (**expi=cisi**). There are also the three hyperbolic forms for divergent hyperbolic double cross, that for $k \times k \times$ (the k -hat double cross), $j \times j \times$ (the j -hat double cross), and $i \times i \times$ (the i -hat double cross). Applying **cisk**, **ciskj**, or **cisi** to the **scleronomic rectangular system** then results in the **rheonomic systems of cylindrical coordinates**. There are then the three forms of complex number and consequently the three derived polar exponentials **expr**, **expq**, and **expz** and three forms Euler's formula for these exponentials: Euler's formula for $r \times$ (**expr=cisr**), Euler's formula for $\theta \times$ (**expq=cisq**), and Euler's formula for $z \times$ (**expz=cisz**), as well as the basis double crosses for hyperbolics. Another application of a **cisk**, **cisj**, **cisi**, scaling/rotating one of the other mutually perpendicular coordinate planes other than the first scaling/rotation, then results in the spherical systems. Each derived basis system resulting from applications of basis exponentials are **cross-product systems**, with difference from the standard rectangular system that the unit vector bases are not single component if expressed in terms of the standard basis. For the spherical system with two consecutive cis-scalings, there are another three forms of complex number, and all of the basis functions, most notably the basis exponentials and Euler's formula similarly as for cylindrical system: Euler's formula for $r \times$ (**expr=cisr**), Euler's formula for $\theta \times$ (**expq=cisq**), and Euler's formula for $\phi \times$ (**expf=cisf**). as well as the basis double cross functions for hyperbolic functions and reflections. All together then, there are 9 forms of Euler's formula: 3 for the standard rectangular system (for k -cross, j -cross, and i -cross), 3 for the cylindrical system (for r -cross, θ -cross, and z -cross), and 3 for the spherical system (for r -cross, θ -cross, and ϕ -cross). This makes 3 scleronomic forms of Euler's formula and 6 rheonomic forms [other less common basis systems also, the most obvious being toroidal... still have to work out how translation of axes fits into the scheme of things especially for robotic arms].

So for the rheonomic cylindrical basis, the $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat basis exponentials are

$$e^{t\hat{r} \times} = 1 + t(\hat{r} \times) - \frac{1}{2!}t^2 - \frac{1}{3!}t^3(\hat{r} \times) + \frac{1}{4!}t^4 + \dots$$

$$e^{t\hat{\theta} \times} = 1 + t(\hat{\theta} \times) - \frac{1}{2!}t^2 - \frac{1}{3!}t^3(\hat{\theta} \times) + \frac{1}{4!}t^4 + \dots$$

$$e^{t\hat{z} \times} = 1 + t(\hat{z} \times) - \frac{1}{2!}t^2 - \frac{1}{3!}t^3(\hat{z} \times) + \frac{1}{4!}t^4 + \dots$$

and hence [with suggested pronunciation]

$$\begin{aligned} \exp(t\hat{r} \times (\cdot)) &= e^{t\hat{r} \times} = \cos r t + (\sin r t) \hat{r} \times && \text{("cis-tee-radius-cross equals cos-radius-tee plus sine-radius-tee radius-cross")} \\ \exp(t\hat{\theta} \times (\cdot)) &= e^{t\hat{\theta} \times} = \cos \theta t + (\sin \theta t) \hat{\theta} \times && \text{("cis-tee-azimuth-cross equals cos-azimuth-tee plus sine-azimuth-tee azimuth-cross")} \\ \exp(t\hat{z} \times (\cdot)) &= e^{t\hat{z} \times} = \cos z t + (\sin z t) \hat{z} \times && \text{("cis-tee-axis-cross equals cos-axis-tee plus sine-axis-tee axis-cross")} \end{aligned}$$

These can be diadically expanded in the polar basis components. Which I will discuss shortly... I'll discuss it now actually; the diadic expansion is seen at sight by placing the component directions straight into the respective columns [with the ordering of the **cross product triangle** for that basis system, here $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat]. This is a direct consequence of, in the polar context, $\hat{r}'\hat{r}=1$, $\hat{\theta}'\hat{\theta}=1$, and $\hat{z}'\hat{z}=1$. For this then, the component directions are required:

For the **$\hat{r}$ -hat exponential**

$$\begin{aligned} \hat{r}\text{-hat direction} & \exp(t\hat{r} \times (\cdot))\hat{r} = e^{t\hat{r} \times}\hat{r} = (\cos r t)\hat{r} + (\sin r t)\hat{r} \times \hat{r} = \hat{r} \\ \hat{\theta}\text{-hat direction} & \exp(t\hat{r} \times (\cdot))\hat{\theta} = e^{t\hat{r} \times}\hat{\theta} = (\cos r t)\hat{\theta} + (\sin r t)\hat{r} \times \hat{\theta} = \hat{\theta} \cos t + \hat{z} \sin t \\ \hat{z}\text{-hat direction} & \exp(t\hat{r} \times (\cdot))\hat{z} = e^{t\hat{r} \times}\hat{z} = (\cos r t)\hat{z} + (\sin r t)\hat{r} \times \hat{z} = \hat{z} \cos t - \hat{\theta} \sin t \end{aligned}$$

For the **$\hat{\theta}$ -hat exponential**

$$\begin{aligned} \hat{r}\text{-hat direction} & \exp(t\hat{\theta} \times (\cdot))\hat{r} = e^{t\hat{\theta} \times}\hat{r} = (\cos \theta t)\hat{r} + (\sin \theta t)\hat{\theta} \times \hat{r} = \hat{r} \cos t - \hat{z} \sin t \\ \hat{\theta}\text{-hat direction} & \exp(t\hat{\theta} \times (\cdot))\hat{\theta} = e^{t\hat{\theta} \times}\hat{\theta} = (\cos \theta t)\hat{\theta} + (\sin \theta t)\hat{\theta} \times \hat{\theta} = \hat{\theta} \\ \hat{z}\text{-hat direction} & \exp(t\hat{\theta} \times (\cdot))\hat{z} = e^{t\hat{\theta} \times}\hat{z} = (\cos \theta t)\hat{z} + (\sin \theta t)\hat{\theta} \times \hat{z} = \hat{z} \cos t + \hat{r} \sin t \end{aligned}$$

For the **$\hat{z}$ -hat exponential**

$$\begin{aligned} \hat{r}\text{-hat direction} & \exp(t\hat{z} \times (\cdot))\hat{r} = e^{t\hat{z} \times}\hat{r} = (\cos z t)\hat{r} + (\sin z t)\hat{z} \times \hat{r} = \hat{r} \cos t + \hat{\theta} \sin t \\ \hat{\theta}\text{-hat direction} & \exp(t\hat{z} \times (\cdot))\hat{\theta} = e^{t\hat{z} \times}\hat{\theta} = (\cos z t)\hat{\theta} + (\sin z t)\hat{z} \times \hat{\theta} = \hat{\theta} \cos t - \hat{r} \sin t \\ \hat{z}\text{-hat direction} & \exp(t\hat{z} \times (\cdot))\hat{z} = e^{t\hat{z} \times}\hat{z} = (\cos z t)\hat{z} + (\sin z t)\hat{z} \times \hat{z} = \hat{z} \end{aligned}$$

for the basis system $\hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0$, $\hat{\theta} = -\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0$, and $\hat{z} = \hat{i} 0 + \hat{j} 0 + \hat{k} 1$.

The inner product of each of cylindrical basis components with itself is unity so that $\hat{r}'\hat{r}=1$, $\hat{\theta}'\hat{\theta}=1$, and $\hat{z}'\hat{z}=1$ similarly as for $\hat{i}'\hat{i}$, $\hat{j}'\hat{j}$, and $\hat{k}'\hat{k}$ with the mixed inner products being zero as required for the basis to remain orthogonal. With $\hat{r}'\hat{r}=1$, $\hat{\theta}'\hat{\theta}=1$ and $\hat{z}'\hat{z}=1$, the above components **miraculously expand diadically** with respect to $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat **into the polar matrix basis exponentials** (**expr**, **expq**, and **expz**) and have EXACTLY the same form [important] as those of the standard rectangular basis exponentials (**expi**, **expj**, and **expk**) but in the polar matrix basis. Notably, the components of each of the directions diadically expand

into the columns of the polar matrix basis exponentials, similarly as those for **expi**, **expj**, and **expk**. Extremely pleased about this. The polar exponential basis functions (**expr**, **expq**, and **expz**) are then the rotation matrices for the $\hat{r}$ -axis, $\hat{\theta}$ -axis, and $\hat{z}$ -axis respectively. This is again a consequence of the properties of the cross product triangle and why the notion of generalized coordinates work in Lagrangian theory. This means that the rotation matrices can themselves be applied to a rotating system independently of the rotation of the basis itself, when for example, the derived basis is rheonomic spinning at a given frequency ω . These can further be expanded out in terms of the standard rectangular system. When expressed in terms of the standard rectangular basis, the polar matrix basis expands into a set of nine matrices that scale the components of the rectangular basis with the sines and cosines from the **expk** scaling applied to the standard rectangular basis to obtain the polar basis. This is hard going mathematically and I will discuss an easier method [with a programmable calculator] after I do all of that maths [in a few pages time]. It works out beautifully... I was not sure if it would. Very pleased that it does.

Note that the elements of the 3by3 polar matrix basis components are not single component 3by3s as for the standard matrix basis except for the $\hat{z}\hat{z}'$ component which is equivalent to the $\hat{k}\hat{k}'$ in the rectangular system. This is one way I have used to try and ensure that the theory that is being developed here is legitimate and works as it should. Since the $\hat{z}$ -hat basis exponential (**expz**) is equivalent to the $\hat{k}$ -hat basis exponential (**expk**), I have diadically expanded **expz** in the polar basis (which has the same form as **expk** straight out), then expanded out the polar basis in terms of the scaled rectangular basis, and demonstrated that it retains its form so that they are equivalent, that is, **expz** is equivalent to **expk** as it should be [but needed to be proved mathematically].

Since the sine and cosine scale factors of cylindrical basis components $\hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta$, $\hat{\theta} = \hat{j} \cos \theta - \hat{i} \sin \theta$, and $\hat{z} = \hat{k}$ are scalar, $\hat{r}$ -cross, $\hat{\theta}$ -cross, and $\hat{z}$ -cross can be written as $\hat{r} \times = (\cos \theta) \hat{i} \times + (\sin \theta) \hat{j} \times$, $\hat{\theta} \times = (\cos \theta) \hat{j} \times - (\sin \theta) \hat{i} \times$, and $\hat{z} \times = \hat{k} \times$. Similarly, the polar basis exponentials can be written in rectangular form as

$$\exp(t \hat{r} \times (\cdot)) = \exp((t \cos \theta) \hat{i} \times + (t \sin \theta) \hat{j} \times) = e^{(t \cos \theta) \hat{i} \times + (t \sin \theta) \hat{j} \times}$$

$$\exp(t \hat{\theta} \times (\cdot)) = \exp((t \cos \theta) \hat{j} \times - (t \sin \theta) \hat{i} \times) = e^{(t \cos \theta) \hat{j} \times - (t \sin \theta) \hat{i} \times}$$

$$\exp(t \hat{z} \times (\cdot)) = \exp(t \hat{k} \times) = e^{t \hat{k} \times} = \cos t + (\sin t) \hat{k} \times \quad \text{the z-hat exponential is exactly the same as the k-hat exponential}$$

and because the $\hat{z}$ -hat basis exponential is exactly the same as the $\hat{k}$ -hat basis exponential, the complex number and complex exponential for the $\hat{r}$ - $\hat{\theta}$ plane is exactly the same as that for the x - y plane. Although the above polar basis exponentials are expressed in terms of $\hat{i} \times$, $\hat{j} \times$, and $\hat{k} \times$, these should only operate on $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat (or these expressed in terms of $\hat{i}$ -hat, $\hat{j}$ -hat, and $\hat{k}$ -hat) because all of this mathematics relies on the properties of the relevant **cross-product triangle**.

De-Moivre's theorem also obviously applies to the cylindrical basis exponentials for powers, and hence frequency scalings, of the $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat basis exponentials

$$(e^{t \hat{r} \times})^n = \exp(nt \hat{r} \times (\cdot)) = \cos r nt + (\sin r nt) \hat{r} \times$$

$$(e^{t \hat{\theta} \times})^n = \exp(nt \hat{\theta} \times (\cdot)) = \cos \theta nt + (\sin \theta nt) \hat{\theta} \times$$

$$(e^{t \hat{z} \times})^n = \exp(nt \hat{z} \times (\cdot)) = \cos z nt + (\sin z nt) \hat{z} \times$$

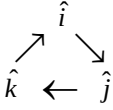
From here, I make a short summary of the important basis system cis-scalings and complex-number like operators, then derive the polar basis functions according to complex theory but with the basis-cross products [that is, by series expansions] and also diadically expand the basis functions into polar matrix bases, then in the following section expand out the polar matrix basis functions in terms of the standard rectangular system, and then provide a change of basis system consistent with staple applied linear algebra that allows all of the complicated mathematics to be done easily in matrices by a programmable calculator.

Summary of cylindrical basis-hats, basis-crosses [basis-cross-products], and functions derived from basis-crosses

The most important results of this document can now be summarised as

rectangular complex numbers	cylindrical complex numbers	spherical complex numbers
$(\hat{k}\times)^2(\cdot) = -1(\cdot)$	$(\hat{r}\times)^2(\cdot) = -1(\cdot)$	$(\hat{r}\times)^2(\cdot) = -1(\cdot)$
$(\hat{j}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\theta}\times)^2(\cdot) = -1(\cdot)$
$(\hat{i}\times)^2(\cdot) = -1(\cdot)$	$(\hat{z}\times)^2(\cdot) = -1(\cdot)$	$(\hat{\phi}\times)^2(\cdot) = -1(\cdot)$

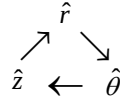
scleronomic
rectangular



rectangular complex
numbers

$\longrightarrow$
cisk θ

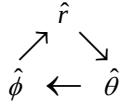
rheonomic
cylindrical



cylindrical complex
numbers

$\longrightarrow$
cisj ϕ

rheonomic
spherical



spherical complex
numbers

Spherical coordinate systems: ciskj-cisk-scaling the scleronomic rectangular basis

The arguments for the spherical system are similar. Transforming the rectangular position variables, that scale the rectangular basis, as spherical polars...

$$\begin{aligned}\vec{r} &= \hat{i} r \sin \theta \cos \phi + \hat{j} r \sin \theta \sin \phi + \hat{k} r \cos \theta \\ &= (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)'\end{aligned}$$

$$\vec{r} = \begin{pmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} r$$

$$e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} \left(\begin{array}{c} \hat{i} \\ \nearrow \quad \searrow \\ \hat{k} \quad \leftarrow \quad \hat{j} \end{array} \right) = e^{\phi \hat{j} \times} \left(\begin{array}{c} e^{\theta \hat{k} \times} \hat{i} \\ \nearrow \quad \searrow \\ e^{\theta \hat{k} \times} \hat{k} \quad \leftarrow \quad e^{\theta \hat{k} \times} \hat{j} \end{array} \right) = \begin{array}{c} e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} \hat{i} \\ \nearrow \quad \searrow \\ e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} \hat{k} \quad \leftarrow \quad e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} \hat{j} \end{array} = \begin{array}{c} \hat{r} \\ \nearrow \quad \searrow \\ \hat{\phi} \quad \leftarrow \quad \hat{\theta} \end{array}$$

note that this is parameterised in terms of magnitude and two phases, one for azimuth, one for colatitude.

Spherical complex numbers: the r -cross, θ -cross, and ϕ -cross rotation operators

Spherical harmonics: the r -hat, θ -hat, and ϕ -hat sines and cosines

Figure 2 Summary: spherical r -hat, θ -hat, and ϕ -hat sines and cosines

Scalar function	r -hat operators	θ -hat operators	ϕ -hat operators
$\sin t$	$(\mathbf{sin} r \ t) \ \hat{r} = \hat{r} t$	$(\mathbf{sin} \theta \ t) \ \hat{r} = \hat{r} \sin t$	$(\mathbf{sin} \phi \ t) \ \hat{r} = \hat{r} \sin t$
	$(\mathbf{sin} r \ t) \ \hat{\theta} = \hat{\theta} \sin t$	$(\mathbf{sin} \theta \ t) \ \hat{\theta} = \hat{\theta} t$	$(\mathbf{sin} \phi \ t) \ \hat{\theta} = \hat{\theta} \sin t$
	$(\mathbf{sin} r \ t) \ \hat{\phi} = \hat{\phi} \sin t$	$(\mathbf{sin} \theta \ t) \ \hat{\phi} = \hat{\phi} \sin t$	$(\mathbf{sin} \phi \ t) \ \hat{\phi} = \hat{\phi} t$
$\cos t$	$(\mathbf{cos} r \ t) \ \hat{r} = \hat{r}$	$(\mathbf{cos} \theta \ t) \ \hat{r} = \hat{r} \cos \theta$	$(\mathbf{cos} \phi \ t) \ \hat{r} = \hat{r} \cos t$
	$(\mathbf{cos} r \ t) \ \hat{\theta} = \hat{\theta} \cos \theta$	$(\mathbf{cos} \theta \ t) \ \hat{\theta} = \hat{\theta}$	$(\mathbf{cos} \phi \ t) \ \hat{\theta} = \hat{\theta} \cos t$
	$(\mathbf{cos} r \ t) \ \hat{\phi} = \hat{\phi} \cos \theta$	$(\mathbf{cos} \theta \ t) \ \hat{\phi} = \hat{\phi} \cos t$	$(\mathbf{cos} \phi \ t) \ \hat{\phi} = \hat{\phi}$

Spherical exponentials: the r -hat, θ -hat, and ϕ -hat exponentials

$$\exp(t \hat{r} \times (\cdot)) = e^{t \hat{r} \times} = \mathbf{cos} r \ t + (\mathbf{sin} r \ t) \hat{r} \times$$

$$\exp(t \hat{\theta} \times (\cdot)) = e^{t \hat{\theta} \times} = \mathbf{cos} \theta \ t + (\mathbf{sin} \theta \ t) \hat{\theta} \times$$

$$\exp(t \hat{\phi} \times (\cdot)) = e^{t \hat{\phi} \times} = \mathbf{cos} \phi \ t + (\mathbf{sin} \phi \ t) \hat{\phi} \times$$

SECTION 2 – COMPLEX NUMBER MATHEMATICS OF THE BASIS FUNCTIONS

POLAR CYLINDRICAL $\hat{r} \times (\cdot)$ BASIS FUNCTIONS: cosr , sinr , expr , secr , and tanr

cylindrical r -hat complex numbers, $\hat{r} \times (\cdot)$, (cyclic group properties) for **sinr , cosr , expr , secr , tanr**

$(\hat{r} \times)^2 (\cdot) = -1(\cdot)$		
r -cross operating on r -hat (degenerate group of order 1)	r -cross operating on θ -hat (cyclic group of order 4)	r -cross operating on z -hat (cyclic group of order 4)
$(\hat{r} \times) \hat{r} = 0 \hat{r}$	$(\hat{r} \times) \hat{\theta} = +1 \hat{z}$	$(\hat{r} \times) \hat{z} = -1 \hat{\theta}$
$(\hat{r} \times)^2 \hat{r} = 0 \hat{r}$	$(\hat{r} \times)^2 \hat{\theta} = -1 \hat{\theta}$	$(\hat{r} \times)^2 \hat{z} = -1 \hat{z}$
$(\hat{r} \times)^3 \hat{r} = 0 \hat{r}$	$(\hat{r} \times)^3 \hat{\theta} = -1 \hat{z}$	$(\hat{r} \times)^3 \hat{z} = +1 \hat{\theta}$
$(\hat{r} \times)^4 \hat{r} = 0 \hat{r}$	$(\hat{r} \times)^4 \hat{\theta} = +1 \hat{\theta}$	$(\hat{r} \times)^4 \hat{z} = +1 \hat{z}$

vector r -hat basis cosine: cosr

$$\begin{aligned}
 \text{cosr } t &= \frac{1}{2} (\exp(t \hat{r} \times) + \exp(-t \hat{r} \times)) \\
 &= \frac{1}{2} \exp(t \hat{r} \times) + \frac{1}{2} \exp(-t \hat{r} \times) \\
 &= \frac{1}{2} \left(1 + t \hat{r} \times + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) \\
 &\quad + \frac{1}{2} \left(1 - t \hat{r} \times + \frac{1}{2!} t^2 (\hat{r} \times)^2 - \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{4!} t^4 (\hat{r} \times)^4 - \dots \right) \\
 &= 1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots
 \end{aligned}$$

θ -hat direction

z -hat direction

r -hat direction

$$\begin{aligned}
 \text{cosr } t \hat{\theta} &= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) \hat{\theta} \\
 &= \hat{\theta} + \frac{1}{2!} t^2 (\hat{r} \times)^2 \hat{\theta} + \frac{1}{4!} t^4 (\hat{r} \times)^4 \hat{\theta} + \dots \\
 &= \hat{\theta} - \frac{1}{2!} t^2 \hat{\theta} + \frac{1}{4!} t^4 \hat{\theta} - \dots \\
 &= \hat{\theta} \left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \dots \right) \\
 &= \hat{\theta} \cos t \\
 \text{cosr } t \hat{z} &= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) \hat{z} \\
 &= \hat{z} + \frac{1}{2!} t^2 (\hat{r} \times)^2 \hat{z} + \frac{1}{4!} t^4 (\hat{r} \times)^4 \hat{z} + \dots \\
 &= \hat{z} - \frac{1}{2!} t^2 \hat{z} + \frac{1}{4!} t^4 \hat{z} - \dots \\
 &= \hat{z} \left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \dots \right) \\
 &= \hat{z} \cos t \\
 \text{cosr } t \hat{r} &= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) \hat{r} \\
 &= \hat{r} + \frac{1}{2!} t^2 (\hat{r} \times)^2 \hat{r} + \frac{1}{4!} t^4 (\hat{r} \times)^4 \hat{r} + \dots \\
 &= \hat{r} - 0 \hat{r} + 0 \hat{r} - \dots \\
 &= \hat{r} (1 + 0 + 0 + \dots) \\
 &= \hat{r}
 \end{aligned}$$

where $(\hat{r} \times)^2 \hat{\theta} = -\hat{\theta}$ and $(\hat{r} \times)^4 \hat{\theta} = \hat{\theta}$ where $(\hat{r} \times)^2 \hat{z} = -\hat{z}$ and $(\hat{r} \times)^4 \hat{z} = \hat{z}$ where $(\hat{r} \times)^2 \hat{r} = 0 \hat{r}$ and $(\hat{r} \times)^4 \hat{r} = 0 \hat{r}$

summary of components: r -hat direction **$\text{cosr } t \hat{r} = \hat{r}$**
 θ -hat direction **$\text{cosr } t \hat{\theta} = \hat{\theta} \cos t$**
 z -hat direction **$\text{cosr } t \hat{z} = \hat{z} \cos t$**

diadic expansion

$$(\text{cosr } t)(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \cos t \end{pmatrix} \underset{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r})1 & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})\cos t & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\cos t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \hat{r}\hat{r}' + \hat{\theta}\hat{\theta}' \cos t + \hat{z}\hat{z}' \cos t$$

vector r -hat basis sine: sinr

$$\begin{aligned}
 (\text{sinr } t) \hat{r} \times &= \frac{1}{2} (\exp(t \hat{r} \times) - \exp(-t \hat{r} \times)) \\
 &= \frac{1}{2} \exp(t \hat{r} \times) - \frac{1}{2} \exp(-t \hat{r} \times) \\
 &= \frac{1}{2} \left(1 + t \hat{r} \times + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) \\
 &\quad - \frac{1}{2} \left(1 - t \hat{r} \times + \frac{1}{2!} t^2 (\hat{r} \times)^2 - \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{4!} t^4 (\hat{r} \times)^4 - \dots \right) \\
 &= t \hat{r} \times + \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{5!} t^5 (\hat{r} \times)^5 + \dots \\
 \text{sinr } t &= t + \frac{1}{3!} t^3 (\hat{r} \times)^2 + \frac{1}{5!} t^5 (\hat{r} \times)^4 + \dots
 \end{aligned}$$

θ -hat direction

z-hat direction

r-hat direction

$$\begin{aligned}
\mathbf{sinr} \, t \, \hat{\theta} &= \left(t + \frac{1}{3!} t^3 (\hat{r} \times)^2 + \frac{1}{5!} t^5 (\hat{r} \times)^4 + \dots \right) \hat{\theta} \\
&= t \hat{\theta} + \frac{1}{3!} t^3 (\hat{r} \times)^2 \hat{\theta} + \frac{1}{5!} t^5 (\hat{r} \times)^4 \hat{\theta} + \dots \\
&= t \hat{\theta} - \frac{1}{3!} t^3 \hat{\theta} + \frac{1}{5!} t^5 \hat{\theta} - \dots \\
&= \hat{\theta} \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \dots \right) \\
&= \hat{\theta} \sin t
\end{aligned}$$

$$\begin{aligned}
\mathbf{sinr} \, t \, \hat{z} &= \left(t + \frac{1}{3!} t^3 (\hat{r} \times)^2 + \frac{1}{5!} t^5 (\hat{r} \times)^4 + \dots \right) \hat{z} \\
&= t \hat{z} + \frac{1}{3!} t^3 (\hat{r} \times)^2 \hat{z} + \frac{1}{5!} t^5 (\hat{r} \times)^4 \hat{z} + \dots \\
&= t \hat{z} - \frac{1}{3!} t^3 \hat{z} + \frac{1}{5!} t^5 \hat{z} - \dots \\
&= \hat{z} \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \dots \right) \\
&= \hat{z} \sin t
\end{aligned}$$

$$\begin{aligned}
\mathbf{sinr} \, t \, \hat{r} &= \left(t + \frac{1}{3!} t^3 (\hat{r} \times)^2 + \frac{1}{5!} t^5 (\hat{r} \times)^4 + \dots \right) \hat{r} \\
&= t \hat{r} + \frac{1}{3!} t^3 (\hat{r} \times)^2 \hat{r} + \frac{1}{5!} t^5 (\hat{r} \times)^4 \hat{r} + \dots \\
&= t \hat{r} - 0 \hat{r} + 0 \hat{r} - \dots \\
&= \hat{r} (t + 0 + 0 + \dots) \\
&= \hat{r} t
\end{aligned}$$

where $(\hat{r} \times)^2 \hat{\theta} = -\hat{\theta}$ and $(\hat{r} \times)^4 \hat{\theta} = \hat{\theta}$

where $(\hat{r} \times)^2 \hat{z} = -\hat{z}$ and $(\hat{r} \times)^4 \hat{z} = \hat{z}$

where $(\hat{r} \times)^2 \hat{r} = -0 \hat{r}$ and $(\hat{r} \times)^4 \hat{r} = 0 \hat{r}$

summary of components: r-hat direction $\mathbf{sinr} \, t \, \hat{r} = \hat{r} t$
 θ -hat direction $\mathbf{sinr} \, t \, \hat{\theta} = \hat{\theta} \sin t$
z-hat direction $\mathbf{sinr} \, t \, \hat{z} = \hat{z} \sin t$

diadic expansion

$$(\mathbf{sinr} \, t)(\cdot) = \begin{pmatrix} t & 0 & 0 \\ 0 & \sin t & 0 \\ 0 & 0 & \sin t \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\sin t & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')\sin t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\hat{r}\hat{r}'t + \hat{\theta}\hat{\theta}'\sin t + \hat{z}\hat{z}'\sin t) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$(\mathbf{sinr} \, t) \hat{r} \times (\cdot) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin t \\ 0 & \sin t & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')0 & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')0 & (\hat{\theta}\hat{z}')(-\sin t) \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')\sin t & (\hat{z}\hat{z}')0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\hat{z}\hat{\theta}' - \hat{\theta}\hat{z}') \sin t$$

vector r-hat basis exponential and Euler's theorem: $\mathbf{expr}$ and $\mathbf{expr} = \mathbf{cistr}$

$$\begin{aligned}
\mathbf{expr} \, t &= \exp(t \hat{r} \times) \\
&= 1 + t \hat{r} \times + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \\
&= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) + \left(t \hat{r} \times + \frac{1}{3!} t^3 (\hat{r} \times)^3 + \frac{1}{5!} t^5 (\hat{r} \times)^5 + \dots \right) \\
&= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} t^4 (\hat{r} \times)^4 + \dots \right) + \left(t + \frac{1}{3!} t^3 (\hat{r} \times)^2 + \frac{1}{5!} t^5 (\hat{r} \times)^4 + \dots \right) \hat{r} \times \\
&= \mathbf{cosr} \, t + (\mathbf{sinr} \, t) \hat{r} \times \\
&= \mathbf{cistr} \, t
\end{aligned}$$

θ -hat direction

z-hat direction

r-hat direction

$$\begin{aligned}
(\mathbf{expr} \, t) \hat{\theta} &= \exp(t \hat{r} \times) \hat{\theta} \\
&= \mathbf{cosr} \, t \, \hat{\theta} + (\mathbf{sinr} \, t) \hat{r} \times \hat{\theta} \\
&= \mathbf{cosr} \, t \, \hat{\theta} + \mathbf{sinr} \, t \, \hat{z} \\
&= \hat{\theta} \cos t + \hat{z} \sin t
\end{aligned}$$

$$\begin{aligned}
(\mathbf{expr} \, t) \hat{z} &= \exp(t \hat{r} \times) \hat{z} \\
&= \mathbf{cosr} \, t \, \hat{z} + (\mathbf{sinr} \, t) \hat{r} \times \hat{z} \\
&= \mathbf{cosr} \, t \, \hat{z} - \mathbf{sinr} \, t \, \hat{\theta} \\
&= \hat{\theta} (-\sin t) + \hat{z} \cos t
\end{aligned}$$

$$\begin{aligned}
(\mathbf{expr} \, t) \hat{r} &= \exp(t \hat{r} \times) \hat{r} \\
&= \mathbf{cosr} \, t \, \hat{r} + (\mathbf{sinr} \, t) \hat{r} \times \hat{r} \\
&= \mathbf{cosr} \, t \, \hat{r} + 0 \hat{r} \\
&= \hat{r}
\end{aligned}$$

summary of components: r-hat direction $(\mathbf{expr} \, t) \hat{r} = \hat{r}$
 θ -hat direction $(\mathbf{expr} \, t) \hat{\theta} = \hat{\theta} \cos t + \hat{z} \sin t$
z-hat direction $(\mathbf{expr} \, t) \hat{z} = \hat{\theta} (-\sin t) + \hat{z} \cos t$

diadic expansion

$$(\mathbf{expr} \, t)(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')1 & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\cos t & -(\hat{\theta}\hat{z}')\sin t \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')\sin t & (\hat{z}\hat{z}')\cos t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \hat{r}\hat{r}' + \hat{\theta}\hat{\theta}'\cos t - \hat{\theta}\hat{z}'\sin t + \hat{z}\hat{\theta}'\sin t + \hat{z}\hat{z}'\cos t$$

de-Moivre's theorem (frequency scaling cisr)

$$\begin{aligned}
 (\text{expr } t)^\omega &= (\exp(t \hat{r} \times))^\omega \\
 &= \exp(\omega t \hat{r} \times) \\
 &= 1 + \omega t \hat{r} \times + \frac{1}{2!} (\omega t)^2 (\hat{r} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{r} \times)^3 + \frac{1}{4!} (\omega t)^4 (\hat{r} \times)^4 + \dots \\
 &= \left(1 + \frac{1}{2!} t^2 (\hat{r} \times)^2 + \frac{1}{4!} (\omega t)^4 (\hat{r} \times)^4 + \dots\right) + \left((\omega t) \hat{r} \times + \frac{1}{3!} (\omega t)^3 (\hat{r} \times)^3 + \frac{1}{5!} (\omega t)^5 (\hat{r} \times)^5 + \dots\right) \\
 &= \left(1 + \frac{1}{2!} (\omega t)^2 (\hat{r} \times)^2 + \frac{1}{4!} (\omega t)^4 (\hat{r} \times)^4 + \dots\right) + \left(\omega t + \frac{1}{3!} (\omega t)^3 (\hat{r} \times)^2 + \frac{1}{5!} (\omega t)^5 (\hat{r} \times)^4 + \dots\right) \hat{r} \times \\
 &= \text{cosr } \omega t + (\text{sinr } \omega t) \hat{r} \times \\
 &= \text{cisr } \omega t
 \end{aligned}$$

vector $\hat{r}$ -basis secant

$$(\text{secr } t)(\cdot) = (\text{cosr } t)^{-1}(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \cos t \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sec t & 0 \\ 0 & 0 & \sec t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r})1 & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})\sec t & -(\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\sec t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector $\hat{r}$ -basis tangent

$$(\text{tanr } t) \hat{r} \times (\cdot) = (\text{cosr } t)^{-1} (\text{sinr } t) \hat{r} \times (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sec t & 0 \\ 0 & 0 & \sec t \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin t \\ 0 & \sin t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan t \\ 0 & \tan t & 0 \end{pmatrix} = (\hat{z}\hat{\theta}' - \hat{\theta}\hat{z}') \tan t$$

POLAR CYLINDRICAL $\hat{\theta} \times (\cdot)$ BASIS FUNCTIONS: cosq, sinq, expq, secq, and tanq

cylindrical θ -hat complex numbers, $\hat{\theta} \times (\cdot)$, (cyclic group properties) for **sinq, cosq, expq, secq, tanq**

$(\hat{\theta} \times)^2(\cdot) = -1(\cdot)$		
θ -cross operating on $\hat{r}$ -hat (cyclic group of order 4)	θ -cross operating on $\hat{\theta}$ -hat (degenerate group of order 1)	θ -cross operating on $\hat{z}$ -hat (cyclic group of order 4)
$(\hat{\theta} \times) \hat{r} = -1\hat{z}$	$(\hat{\theta} \times) \hat{\theta} = 0\hat{\theta}$	$(\hat{\theta} \times) \hat{z} = +1\hat{r}$
$(\hat{\theta} \times)^2 \hat{r} = -1\hat{r}$	$(\hat{\theta} \times)^2 \hat{\theta} = 0\hat{\theta}$	$(\hat{\theta} \times)^2 \hat{z} = -1\hat{z}$
$(\hat{\theta} \times)^3 \hat{r} = +1\hat{z}$	$(\hat{\theta} \times)^3 \hat{\theta} = 0\hat{\theta}$	$(\hat{\theta} \times)^3 \hat{z} = -1\hat{r}$
$(\hat{\theta} \times)^4 \hat{r} = +1\hat{r}$	$(\hat{\theta} \times)^4 \hat{\theta} = 0\hat{\theta}$	$(\hat{\theta} \times)^4 \hat{z} = +1\hat{z}$

vector θ -hat basis cosine: cosq

$$\begin{aligned}
 \text{cosq } t &= \frac{1}{2} (\exp(t \hat{\theta} \times) + \exp(-t \hat{\theta} \times)) \\
 &= \frac{1}{2} \exp(t \hat{\theta} \times) + \frac{1}{2} \exp(-t \hat{\theta} \times) \\
 &= \frac{1}{2} \left(1 + t \hat{\theta} \times + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots\right) \\
 &\quad + \frac{1}{2} \left(1 - t \hat{\theta} \times + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 - \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 - \dots\right) \\
 &= 1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots
 \end{aligned}$$

$\hat{r}$ -hat direction

$\hat{z}$ -hat direction

θ -hat direction

$$\begin{aligned}
 \text{cosq } t \hat{r} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots\right) \hat{r} & \text{cosq } t \hat{z} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots\right) \hat{z} & \text{cosq } t \hat{\theta} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots\right) \hat{\theta} \\
 &= \hat{r} + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 \hat{r} + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 \hat{r} + \dots & &= \hat{z} + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 \hat{z} + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 \hat{z} + \dots & &= \hat{\theta} + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 \hat{\theta} + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 \hat{\theta} + \dots \\
 &= \hat{r} - \frac{1}{2!} t^2 \hat{r} + \frac{1}{4!} t^4 \hat{r} - \dots & &= \hat{z} - \frac{1}{2!} t^2 \hat{z} + \frac{1}{4!} t^4 \hat{z} - \dots & &= \hat{\theta} - 0\hat{\theta} + 0\hat{\theta} - \dots \\
 &= \hat{r} \left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \dots\right) & &= \hat{z} \left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \dots\right) & &= \hat{\theta} (1 + 0 + 0 + \dots) \\
 &= \hat{r} \cos t & &= \hat{z} \cos t & &= \hat{\theta}
 \end{aligned}$$

where $(\hat{\theta} \times)^2 \hat{r} = -\hat{r}$ and $(\hat{\theta} \times)^4 \hat{r} = \hat{r}$

where $(\hat{\theta} \times)^2 \hat{z} = -\hat{z}$ and $(\hat{\theta} \times)^4 \hat{z} = \hat{z}$

where $(\hat{\theta} \times)^2 \hat{\theta} = -0\hat{\theta}$ and $(\hat{\theta} \times)^4 \hat{\theta} = 0\hat{\theta}$

summary of components: r -hat direction **cosq** $t \hat{r} = \hat{r} \cos t$
 θ -hat direction **cosq** $t \hat{\theta} = \hat{\theta}$
 z -hat direction **cosq** $t \hat{z} = \hat{z} \cos t$

diadic expansion

$$(\mathbf{cosq} \, t)(\cdot) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos t \end{pmatrix} \stackrel{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r})0 \cos t & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})1 & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\cos t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector θ -hat basis sine: **sinq**

$$\begin{aligned} (\mathbf{sinq} \, t) \hat{\theta} \times &= \frac{1}{2} \left(\exp(t \hat{\theta} \times) - \exp(-t \hat{\theta} \times) \right) \\ &= \frac{1}{2} \exp(t \hat{\theta} \times) - \frac{1}{2} \exp(-t \hat{\theta} \times) \\ &= \frac{1}{2} \left(1 + t \hat{\theta} \times + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots \right) \\ &\quad - \frac{1}{2} \left(1 - t \hat{\theta} \times + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 - \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 - \dots \right) \\ &= t \hat{\theta} \times + \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{5!} t^5 (\hat{\theta} \times)^5 + \dots \\ \mathbf{sinq} \, t &= t + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 + \dots \end{aligned}$$

r -hat direction

z -hat direction

θ -hat direction

$$\begin{aligned} \mathbf{sinq} \, t \hat{r} &= \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 + \dots \right) \hat{r} & \mathbf{sinq} \, t \hat{z} &= \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 + \dots \right) \hat{z} & \mathbf{sinq} \, t \hat{\theta} &= \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 + \dots \right) \hat{\theta} \\ &= t \hat{r} + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 \hat{r} + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 \hat{r} + \dots & &= t \hat{z} + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 \hat{z} + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 \hat{z} + \dots & &= t \hat{\theta} + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 \hat{\theta} + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 \hat{\theta} + \dots \\ &= t \hat{r} - \frac{1}{3!} t^3 \hat{r} + \frac{1}{5!} t^5 \hat{r} - \dots & &= t \hat{z} - \frac{1}{3!} t^3 \hat{z} + \frac{1}{5!} t^5 \hat{z} - \dots & &= t \hat{\theta} - 0 \hat{\theta} + 0 \hat{\theta} - \dots \\ &= \hat{r} \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \dots \right) & &= \hat{z} \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \dots \right) & &= \hat{\theta} (t + 0 + 0 + \dots) \\ &= \hat{r} \sin t & &= \hat{z} \sin t & &= \hat{\theta} t \end{aligned}$$

where $(\hat{\theta} \times)^2 \hat{r} = -\hat{r}$ and $(\hat{\theta} \times)^4 \hat{r} = \hat{r}$ where $(\hat{\theta} \times)^2 \hat{z} = -\hat{z}$ and $(\hat{\theta} \times)^4 \hat{z} = \hat{z}$ where $(\hat{\theta} \times)^2 \hat{\theta} = -0 \hat{\theta}$ and $(\hat{\theta} \times)^4 \hat{\theta} = 0 \hat{\theta}$

summary of components: r -hat direction **sinq** $t \hat{r} = \hat{r} \sin t$
 θ -hat direction **sinq** $t \hat{\theta} = \hat{\theta} t$
 z -hat direction **sinq** $t \hat{z} = \hat{z} \sin t$

diadic expansion

$$\begin{aligned} (\mathbf{sinq} \, t)(\cdot) &= \begin{pmatrix} \sin t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & \sin t \end{pmatrix} \stackrel{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r})\sin t & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})t & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\sin t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ (\mathbf{sinq} \, t) \hat{\theta} \times (\cdot) &= \begin{pmatrix} 0 & 0 & \sin t \\ 0 & 0 & 0 \\ -\sin t & 0 & 0 \end{pmatrix} \stackrel{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r})0 & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})\sin t \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})0 & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})(-\sin t) & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \end{aligned}$$

vector θ -hat basis exponential and Euler's theorem: **expq** and **expq** = **cisq**

$$\begin{aligned} \mathbf{expq} \, t &= \exp(t \hat{\theta} \times) \\ &= 1 + t \hat{\theta} \times + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots \right) + \left(t \hat{\theta} \times + \frac{1}{3!} t^3 (\hat{\theta} \times)^3 + \frac{1}{5!} t^5 (\hat{\theta} \times)^5 + \dots \right) \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^2 + \frac{1}{4!} t^4 (\hat{\theta} \times)^4 + \dots \right) + \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^2 + \frac{1}{5!} t^5 (\hat{\theta} \times)^4 + \dots \right) \hat{\theta} \times \\ &= \mathbf{cosq} \, t + (\mathbf{sinq} \, t) \hat{\theta} \times \\ &= \mathbf{cisq} \, t \end{aligned}$$

r-hat direction

$$\begin{aligned}(\mathbf{expq}t)\hat{r} &= \exp(t\hat{\theta}\times)\hat{r} \\ &= \cosqt\hat{r} + (\sinqt)\hat{\theta}\times\hat{r} \\ &= \cosqt\hat{r} - \sinqt\hat{z} \\ &= \hat{r}\cos t - \hat{z}\sin t\end{aligned}$$

z-hat direction

$$\begin{aligned}(\mathbf{expq}t)\hat{z} &= \exp(t\hat{\theta}\times)\hat{z} \\ &= \cosqt\hat{z} + (\sinqt)\hat{\theta}\times\hat{z} \\ &= \cosqt\hat{z} + \sinqt\hat{r} \\ &= \hat{r}\sin t + \hat{z}\cos t\end{aligned}$$

θ -hat direction

$$\begin{aligned}(\mathbf{expq}t)\hat{\theta} &= \exp(t\hat{\theta}\times)\hat{\theta} \\ &= \cosqt\hat{\theta} + (\sinqt)\hat{\theta}\times\hat{\theta} \\ &= \cosqt\hat{\theta} + 0\hat{\theta} \\ &= \hat{\theta}\end{aligned}$$

summary of components: r -hat direction $(\mathbf{expq}t)\hat{r} = \hat{r}\cos t - \hat{z}\sin t$
 θ -hat direction $(\mathbf{expq}t)\hat{\theta} = \hat{\theta}$
 z -hat direction $(\mathbf{expq}t)\hat{z} = \hat{r}\sin t + \hat{z}\cos t$

diadic expansion

$$(\mathbf{expq}t)(\cdot) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')\cos t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')\sin t \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')1 & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')(-\sin t) & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')\cos t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

de-Moivre's theorem (frequency scaling cisq)

$$\begin{aligned}(\mathbf{expq}t)^\omega &= \left(\exp(t\hat{\theta}\times)\right)^\omega \\ &= \exp(\omega t\hat{\theta}\times) \\ &= 1 + \omega t\hat{\theta}\times + \frac{1}{2!}(\omega t)^2(\hat{\theta}\times)^2 + \frac{1}{3!}(\omega t)^3(\hat{\theta}\times)^3 + \frac{1}{4!}(\omega t)^4(\hat{\theta}\times)^4 + \dots \\ &= \left(1 + \frac{1}{2!}t^2(\hat{\theta}\times)^2 + \frac{1}{4!}(\omega t)^4(\hat{\theta}\times)^4 + \dots\right) + \left((\omega t)\hat{\theta}\times + \frac{1}{3!}(\omega t)^3(\hat{\theta}\times)^3 + \frac{1}{5!}(\omega t)^5(\hat{\theta}\times)^5 + \dots\right) \\ &= \left(1 + \frac{1}{2!}(\omega t)^2(\hat{\theta}\times)^2 + \frac{1}{4!}(\omega t)^4(\hat{\theta}\times)^4 + \dots\right) + \left(\omega t + \frac{1}{3!}(\omega t)^3(\hat{\theta}\times)^2 + \frac{1}{5!}(\omega t)^5(\hat{\theta}\times)^4 + \dots\right)\hat{\theta}\times \\ &= \cosq\omega t + (\sinq\omega t)\hat{\theta}\times \\ &= \text{cisq}\omega t\end{aligned}$$

vector θ -hat basis secant

$$(\mathbf{secq})(\cdot) = (\mathbf{cosq}t)^{-1}(\cdot) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos t \end{pmatrix}_{\text{polar basis}}^{-1} = \begin{pmatrix} \sec t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sec t \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')\sec t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')1 & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')\sec t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector θ -hat basis tangent

$$(\mathbf{tanq}t)\hat{\theta}\times(\cdot) = (\mathbf{cosq}t)^{-1}(\sinqt)\hat{\theta}\times(\cdot) = \begin{pmatrix} \sec t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sec t \end{pmatrix}_{\text{polar basis}} \begin{pmatrix} 0 & 0 & \sin t \\ 0 & 0 & 0 \\ -\sin t & 0 & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} 0 & 0 & \tan t \\ 0 & 0 & 0 \\ -\tan t & 0 & 0 \end{pmatrix}_{\text{polar basis}}$$

POLAR CYLINDRICAL $\hat{z}\times(\cdot)$ BASIS FUNCTIONS: cosz, sinz, expz, secz, and tanz

cylindrical z -hat complex numbers, $\hat{z}\times(\cdot)$, (cyclic group properties) for **sinz, cosz, expz, secz, tanz**

$(\hat{z}\times)^2(\cdot) = -1(\cdot)$		
z-cross operating on r -hat (cyclic group of order 4)	z-cross operating on θ -hat (cyclic group of order 4)	z-cross operating on z -hat (degenerate group of order 1)
$(\hat{z}\times)\hat{z} = 0\hat{z}$	$(\hat{z}\times)\hat{\theta} = -1\hat{r}$	$(\hat{z}\times)\hat{r} = +1\hat{\theta}$
$(\hat{z}\times)^2\hat{z} = 0\hat{z}$	$(\hat{z}\times)^2\hat{\theta} = -1\hat{\theta}$	$(\hat{z}\times)^2\hat{r} = -1\hat{r}$
$(\hat{z}\times)^3\hat{z} = 0\hat{z}$	$(\hat{z}\times)^3\hat{\theta} = +1\hat{r}$	$(\hat{z}\times)^3\hat{r} = -1\hat{\theta}$
$(\hat{z}\times)^4\hat{z} = 0\hat{z}$	$(\hat{z}\times)^4\hat{\theta} = +1\hat{\theta}$	$(\hat{z}\times)^4\hat{r} = +1\hat{r}$

vector z-hat basis cosine: cosz

$$\begin{aligned}
 \cosz t &= \frac{1}{2}(\exp(t \hat{z} \times) + \exp(-t \hat{z} \times)) \\
 &= \frac{1}{2}\exp(t \hat{z} \times) + \frac{1}{2}\exp(-t \hat{z} \times) \\
 &= \frac{1}{2}\left(1 + t \hat{z} \times + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{3!}t^3(\hat{z} \times)^3 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots\right) \\
 &\quad + \frac{1}{2}\left(1 - t \hat{z} \times + \frac{1}{2!}t^2(\hat{z} \times)^2 - \frac{1}{3!}t^3(\hat{z} \times)^3 + \frac{1}{4!}t^4(\hat{z} \times)^4 - \dots\right) \\
 &= 1 + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots
 \end{aligned}$$

r-hat direction

θ -hat direction

z-hat direction

$$\begin{aligned}
 \cosz t \hat{r} &= \left(1 + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots\right) \hat{r} & \cosz t \hat{\theta} &= \left(1 + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots\right) \hat{\theta} & \cosz t \hat{z} &= \left(1 + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots\right) \hat{z} \\
 &= \hat{r} + \frac{1}{2!}t^2(\hat{z} \times)^2 \hat{r} + \frac{1}{4!}t^4(\hat{z} \times)^4 \hat{r} + \dots & &= \hat{\theta} + \frac{1}{2!}t^2(\hat{z} \times)^2 \hat{\theta} + \frac{1}{4!}t^4(\hat{z} \times)^4 \hat{\theta} + \dots & &= \hat{z} + \frac{1}{2!}t^2(\hat{z} \times)^2 \hat{z} + \frac{1}{4!}t^4(\hat{z} \times)^4 \hat{z} + \dots \\
 &= \hat{r} - \frac{1}{2!}t^2 \hat{r} + \frac{1}{4!}t^4 \hat{r} - \dots & &= \hat{\theta} - \frac{1}{2!}t^2 \hat{\theta} + \frac{1}{4!}t^4 \hat{\theta} - \dots & &= \hat{z} - 0 \hat{z} + 0 \hat{z} - \dots \\
 &= \hat{r} \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \dots\right) & &= \hat{\theta} \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \dots\right) & &= \hat{z} (1 + 0 + 0 + \dots) \\
 &= \hat{r} \cos t & &= \hat{\theta} \cos t & &= \hat{z}
 \end{aligned}$$

where $(\hat{z} \times)^2 \hat{r} = -\hat{r}$ and $(\hat{z} \times)^4 \hat{r} = \hat{r}$

where $(\hat{z} \times)^2 \hat{\theta} = -\hat{\theta}$ and $(\hat{z} \times)^4 \hat{\theta} = \hat{\theta}$

where $(\hat{z} \times)^2 \hat{z} = -0 \hat{z}$ and $(\hat{z} \times)^4 \hat{z} = 0 \hat{z}$

summary of components: *r*-hat direction **sinz** $t \hat{r} = \hat{r} \sin t$
 θ -hat direction **sinz** $t \hat{\theta} = \hat{\theta} \sin t$
z-hat direction **sinz** $t \hat{z} = \hat{z} t$

diadic expansion

$$(\cosz t)(\cdot) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \text{polar} \\ \text{basis} \end{matrix} = \begin{pmatrix} (\hat{r}\hat{r}')\cos t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\cos t & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector z-hat basis sine: sinz

$$\begin{aligned}
 (\sinz t) \hat{z} \times &= \frac{1}{2}(\exp(t \hat{z} \times) - \exp(-t \hat{z} \times)) \\
 &= \frac{1}{2}\exp(t \hat{z} \times) - \frac{1}{2}\exp(-t \hat{z} \times) \\
 &= \frac{1}{2}\left(1 + t \hat{z} \times + \frac{1}{2!}t^2(\hat{z} \times)^2 + \frac{1}{3!}t^3(\hat{z} \times)^3 + \frac{1}{4!}t^4(\hat{z} \times)^4 + \dots\right) \\
 &\quad - \frac{1}{2}\left(1 - t \hat{z} \times + \frac{1}{2!}t^2(\hat{z} \times)^2 - \frac{1}{3!}t^3(\hat{z} \times)^3 + \frac{1}{4!}t^4(\hat{z} \times)^4 - \dots\right) \\
 &= t \hat{z} \times + \frac{1}{3!}t^3(\hat{z} \times)^3 + \frac{1}{5!}t^5(\hat{z} \times)^5 + \dots \\
 \sinz t &= t + \frac{1}{3!}t^3(\hat{z} \times)^2 + \frac{1}{5!}t^5(\hat{z} \times)^4 + \dots
 \end{aligned}$$

r-hat direction

θ -hat direction

z-hat direction

$$\begin{aligned}
 \sinz t \hat{r} &= \left(t + \frac{1}{3!}t^3(\hat{z} \times)^2 + \frac{1}{5!}t^5(\hat{z} \times)^4 + \dots\right) \hat{r} & \sinz t \hat{\theta} &= \left(t + \frac{1}{3!}t^3(\hat{z} \times)^2 + \frac{1}{5!}t^5(\hat{z} \times)^4 + \dots\right) \hat{\theta} & \sinz t \hat{z} &= \left(t + \frac{1}{3!}t^3(\hat{z} \times)^2 + \frac{1}{5!}t^5(\hat{z} \times)^4 + \dots\right) \hat{z} \\
 &= t \hat{r} + \frac{1}{3!}t^3(\hat{z} \times)^2 \hat{r} + \frac{1}{5!}t^5(\hat{z} \times)^4 \hat{r} + \dots & &= t \hat{\theta} + \frac{1}{3!}t^3(\hat{z} \times)^2 \hat{\theta} + \frac{1}{5!}t^5(\hat{z} \times)^4 \hat{\theta} + \dots & &= t \hat{z} + \frac{1}{3!}t^3(\hat{z} \times)^4 \hat{z} + \frac{1}{5!}t^5(\hat{z} \times)^4 \hat{z} + \dots \\
 &= t \hat{r} - \frac{1}{3!}t^3 \hat{r} + \frac{1}{5!}t^5 \hat{r} - \dots & &= t \hat{\theta} - \frac{1}{3!}t^3 \hat{\theta} + \frac{1}{5!}t^5 \hat{\theta} - \dots & &= t \hat{z} - 0 \hat{z} + 0 \hat{z} - \dots \\
 &= \hat{r} \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots\right) & &= \hat{\theta} \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots\right) & &= \hat{z} (t + 0 + 0 + \dots) \\
 &= \hat{r} \sin t & &= \hat{\theta} \sin t & &= \hat{z} t
 \end{aligned}$$

where $(\hat{z} \times)^2 \hat{r} = -\hat{r}$ and $(\hat{z} \times)^4 \hat{r} = \hat{r}$

where $(\hat{\theta} \times)^2 \hat{r} = -\hat{r}$ and $(\hat{\theta} \times)^4 \hat{r} = \hat{r}$

where $(\hat{z} \times)^2 \hat{z} = -0 \hat{z}$ and $(\hat{z} \times)^4 \hat{z} = 0 \hat{z}$

summary of components: *r*-hat direction **cosz** $t \hat{r} = \hat{r} \cos t$
 θ -hat direction **cosz** $t \hat{\theta} = \hat{\theta} \cos t$
z-hat direction **cosz** $t \hat{z} = \hat{z}$

diadic expansion

$$(\sinz t)(\cdot) = \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \sin t & 0 \\ 0 & 0 & t \end{pmatrix} \begin{matrix} \text{polar} \\ \text{basis} \end{matrix} = \begin{pmatrix} (\hat{r}\hat{r}')\sin t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\sin t & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$(\mathbf{sinz}t) \hat{z} \times (\cdot) = \begin{pmatrix} 0 & -\sin t & 0 \\ \sin t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')0 & (\hat{r}\hat{\theta}')(-\sin t) & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')\sin t & (\hat{\theta}\hat{\theta}')0 & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector z-hat basis exponential and Euler's theorem: expz and expz = cisz

$$\begin{aligned} \mathbf{expz}t &= \exp(t \hat{z} \times) \\ &= 1 + t \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times)^4 + \dots \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^2 + \frac{1}{4!} t^4 (\hat{z} \times)^4 + \dots\right) + \left(t \hat{z} \times + \frac{1}{3!} t^3 (\hat{z} \times)^3 + \frac{1}{5!} t^5 (\hat{z} \times)^5 + \dots\right) \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^2 + \frac{1}{4!} t^4 (\hat{z} \times)^4 + \dots\right) + \left(t + \frac{1}{3!} t^3 (\hat{z} \times)^2 + \frac{1}{5!} t^5 (\hat{z} \times)^4 + \dots\right) \hat{z} \times \\ &= \mathbf{cosz}t + (\mathbf{sinz}t) \hat{z} \times \\ &= \mathbf{ciszt} \end{aligned}$$

r-hat direction

$$\begin{aligned} (\mathbf{expz}t) \hat{r} &= \exp(t \hat{z} \times) \hat{r} \\ &= \mathbf{cosz}t \hat{r} + (\mathbf{sinz}t) \hat{z} \times \hat{r} \\ &= \mathbf{cosz}t \hat{r} + \mathbf{sinz}t \hat{\theta} \\ &= \hat{r} \cos t + \hat{\theta} \sin t \end{aligned}$$

θ-hat direction

$$\begin{aligned} (\mathbf{expz}t) \hat{\theta} &= \exp(t \hat{z} \times) \hat{\theta} \\ &= \mathbf{cosz}t \hat{\theta} + (\mathbf{sinz}t) \hat{r} \times \hat{\theta} \\ &= \mathbf{cosz}t \hat{\theta} + \mathbf{sinz}t \hat{z} \\ &= \hat{\theta} \cos t + \hat{z} \sin t \end{aligned}$$

z-hat direction

$$\begin{aligned} (\mathbf{expz}t) \hat{z} &= \exp(t \hat{z} \times) \hat{z} \\ &= \mathbf{cosz}t \hat{z} + (\mathbf{sinz}t) \hat{z} \times \hat{z} \\ &= \mathbf{cosz}t \hat{z} + 0 \hat{z} \\ &= \hat{z} \end{aligned}$$

summary of components: r-hat direction $(\mathbf{expz}t) \hat{r} = \hat{r} \cos t + \hat{\theta} \sin t$
θ-hat direction $(\mathbf{expz}t) \hat{\theta} = \hat{\theta} \cos t + \hat{z} \sin t$
z-hat direction $(\mathbf{expz}t) \hat{z} = \hat{z}$

diadic expansion

$$(\mathbf{expz}t)(\cdot) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}') \cos t & (\hat{r}\hat{\theta}')(-\sin t) & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}') \sin t & (\hat{\theta}\hat{\theta}') \cos t & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

de-Moivre's theorem (frequency scaling cisz)

$$\begin{aligned} (\mathbf{expz}t)^\omega &= (\exp(t \hat{z} \times))^\omega \\ &= \exp(\omega t \hat{z} \times) \\ &= 1 + \omega t \hat{z} \times + \frac{1}{2!} (\omega t)^2 (\hat{z} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{z} \times)^3 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times)^4 + \dots \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^2 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times)^4 + \dots\right) + \left((\omega t) \hat{z} \times + \frac{1}{3!} (\omega t)^3 (\hat{z} \times)^3 + \frac{1}{5!} (\omega t)^5 (\hat{z} \times)^5 + \dots\right) \\ &= \left(1 + \frac{1}{2!} (\omega t)^2 (\hat{z} \times)^2 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times)^4 + \dots\right) + \left(\omega t + \frac{1}{3!} (\omega t)^3 (\hat{z} \times)^2 + \frac{1}{5!} (\omega t)^5 (\hat{z} \times)^4 + \dots\right) \hat{z} \times \\ &= \mathbf{cosz} \omega t + (\mathbf{sinz} \omega t) \hat{z} \times \\ &= \mathbf{ciszt} \omega t \end{aligned}$$

vector z-hat basis secant

$$(\mathbf{seczt})(\cdot) = (\mathbf{coszt})^{-1}(\cdot) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}}^{-1} = \begin{pmatrix} \sec t & 0 & 0 \\ 0 & \sec t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}') \sec t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}') \sec t & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector z-hat basis tangent

$$(\mathbf{tanzt}) \hat{z} \times (\cdot) = (\mathbf{coszt})^{-1} (\mathbf{sinzt}) \hat{z} \times (\cdot) = \begin{pmatrix} \sec t & 0 & 0 \\ 0 & \sec t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} \begin{pmatrix} 0 & -\sin t & 0 \\ \sin t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} 0 & -\tan t & 0 \\ \tan t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{polar basis}}$$

POLAR CYLINDRICAL DOUBLE CROSS $\hat{r} \times \hat{r} \times (\cdot)$ BASIS FUNCTIONS: cosrr, sinrr, exprr, secr, and tanrr

vector $\hat{r}$ -hat double cross basis cosine: cosrr

$$\begin{aligned}\cosrr t &= \frac{1}{2}(\exp(t\hat{r} \times \hat{r} \times) + \exp(-t\hat{r} \times \hat{r} \times)) \\ &= \frac{1}{2}\exp(t\hat{r} \times \hat{r} \times) + \frac{1}{2}\exp(-t\hat{r} \times \hat{r} \times) \\ &= \frac{1}{2}\left(1 + t\hat{r} \times \hat{r} \times + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^3 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 + \dots\right) \\ &\quad + \frac{1}{2}\left(1 - t\hat{r} \times \hat{r} \times + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 - \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^3 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 - \dots\right) \\ &= 1 + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 + \dots \\ &= 1 + \frac{1}{2!}t^2(\hat{r} \times)^4 + \frac{1}{4!}t^4(\hat{r} \times)^8 + \dots\end{aligned}$$

θ -hat direction

$$\begin{aligned}\cosrr t \hat{\theta} &= \left(1 + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^4 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^8 + \dots\right)\hat{\theta} \\ &= \hat{\theta} + \frac{1}{2!}t^2(\hat{r} \times)^4\hat{\theta} + \frac{1}{4!}t^4(\hat{r} \times)^8\hat{\theta} + \dots \\ &= \hat{\theta} + \frac{1}{2!}t^2\hat{\theta} + \frac{1}{4!}t^4\hat{\theta} + \dots \\ &= \hat{\theta}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots\right) \\ &= \hat{\theta} \cosh t\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{\theta} = (\hat{r} \times)^8 \hat{\theta} = \hat{\theta}$$

z -hat direction

$$\begin{aligned}\cosrr t \hat{z} &= \left(1 + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 + \dots\right)\hat{z} \\ &= \hat{z} + \frac{1}{2!}t^2(\hat{r} \times)^4\hat{z} + \frac{1}{4!}t^4(\hat{r} \times)^8\hat{z} + \dots \\ &= \hat{z} + \frac{1}{2!}t^2\hat{z} + \frac{1}{4!}t^4\hat{z} + \dots \\ &= \hat{z}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots\right) \\ &= \hat{z} \cosh t\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{z} = (\hat{r} \times)^8 \hat{z} = \hat{z}$$

r -hat direction

$$\begin{aligned}(\cosrr t) \hat{r} &= \left(1 + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 + \dots\right)\hat{r} \\ &= \hat{r} + \frac{1}{2!}t^2(\hat{r} \times)^4\hat{r} + \frac{1}{4!}t^4(\hat{r} \times)^8\hat{r} + \dots \\ &= \hat{r} + 0\hat{r} + 0\hat{r} + \dots \\ &= \hat{r}(1 + 0 + 0 + \dots) \\ &= \hat{r}\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{r} = (\hat{r} \times)^8 \hat{r} = 0\hat{r}$$

summary of components: r -hat direction $\cosrr t \hat{r} = \hat{r}$
 θ -hat direction $\cosrr t \hat{\theta} = \hat{\theta} \cosh t$
 z -hat direction $\cosrr t \hat{z} = \hat{z} \cosh t$

diadic expansion

$$(\cosrr t)(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & 0 \\ 0 & 0 & \cosh t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r})1 & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})\cosh t & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\cosh t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \hat{r}\hat{r}' + \hat{\theta}\hat{\theta}'\cosh t + \hat{z}\hat{z}'\cosh t$$

vector $\hat{r}$ -hat double cross basis sine: sinrr

$$\begin{aligned}(\sinrr t) \hat{r} \times &= \frac{1}{2}(\exp(t\hat{r} \times \hat{r} \times) - \exp(-t\hat{r} \times \hat{r} \times)) \\ &= \frac{1}{2}\exp(t\hat{r} \times \hat{r} \times) - \frac{1}{2}\exp(-t\hat{r} \times \hat{r} \times) \\ &= \frac{1}{2}\left(1 + t(\hat{r} \times)^2 + \frac{1}{2!}t^2(\hat{r} \times)^4 + \frac{1}{3!}t^3(\hat{r} \times)^6 + \frac{1}{4!}t^4(\hat{r} \times)^8 + \dots\right) \\ &\quad - \frac{1}{2}\left(1 - t(\hat{r} \times)^2 + \frac{1}{2!}t^2(\hat{r} \times)^4 - \frac{1}{3!}t^3(\hat{r} \times)^6 + \frac{1}{4!}t^4(\hat{r} \times)^8 - \dots\right) \\ &= t(\hat{r} \times)^2 + \frac{1}{3!}t^3(\hat{r} \times)^6 + \frac{1}{5!}t^5(\hat{r} \times)^{10} + \dots \\ \sinrr t &= t + \frac{1}{3!}t^3(\hat{r} \times)^4 + \frac{1}{5!}t^5(\hat{r} \times)^8 + \dots\end{aligned}$$

θ -hat direction

$$\begin{aligned}\sinrr t \hat{\theta} &= \left(t + \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^2 + \frac{1}{5!}t^5(\hat{r} \times \hat{r} \times)^4 + \dots\right)\hat{\theta} \\ &= \left(t + \frac{1}{3!}t^3(\hat{r} \times)^4 + \frac{1}{5!}t^5(\hat{r} \times)^8 + \dots\right)\hat{\theta} \\ &= t\hat{\theta} + \frac{1}{3!}t^3(\hat{r} \times)^4\hat{\theta} + \frac{1}{5!}t^5(\hat{r} \times)^8\hat{\theta} + \dots \\ &= t\hat{\theta} + \frac{1}{3!}t^3\hat{\theta} + \frac{1}{5!}t^5\hat{\theta} + \dots \\ &= \hat{\theta}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right) \\ &= \hat{\theta} \sinh t\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{\theta} = (\hat{r} \times)^8 \hat{\theta} = \hat{\theta}$$

z -hat direction

$$\begin{aligned}\sinrr t \hat{z} &= \left(t + \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^2 + \frac{1}{5!}t^5(\hat{r} \times \hat{r} \times)^4 + \dots\right)\hat{z} \\ &= \left(t + \frac{1}{3!}t^3(\hat{r} \times)^4 + \frac{1}{5!}t^5(\hat{r} \times)^8 + \dots\right)\hat{z} \\ &= t\hat{z} + \frac{1}{3!}t^3(\hat{r} \times)^4\hat{z} + \frac{1}{5!}t^5(\hat{r} \times)^8\hat{z} + \dots \\ &= t\hat{z} + \frac{1}{3!}t^3\hat{z} + \frac{1}{5!}t^5\hat{z} + \dots \\ &= \hat{z}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right) \\ &= \hat{z} \sinh t\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{z} = (\hat{r} \times)^8 \hat{z} = \hat{z}$$

r -hat direction

$$\begin{aligned}\sinrr t \hat{r} &= \left(t + \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^3 + \frac{1}{5!}t^5(\hat{r} \times \hat{r} \times)^5 + \dots\right)\hat{r} \\ &= \left(t + \frac{1}{3!}t^3(\hat{r} \times)^6 + \frac{1}{5!}t^5(\hat{r} \times)^{10} + \dots\right)\hat{r} \\ &= t\hat{r} + \frac{1}{3!}t^3(\hat{r} \times)^6\hat{r} + \frac{1}{5!}t^5(\hat{r} \times)^{10}\hat{r} + \dots \\ &= t\hat{r} + 0\hat{r} + 0\hat{r} + \dots \\ &= \hat{r}(t + 0 + 0 + \dots) \\ &= \hat{r}t\end{aligned}$$

$$\text{where } (\hat{r} \times)^4 \hat{r} = (\hat{r} \times)^8 \hat{r} = 0\hat{r}$$

summary of components: r -hat direction $\sinrr t \hat{r} = \hat{r}t$

$$\begin{aligned}\theta\text{-hat direction } \mathbf{\hat{\theta}} &= \hat{\theta} \sinh t \\ z\text{-hat direction } \mathbf{\hat{z}} &= \hat{z} \sinh t\end{aligned}$$

diadic expansion

$$\begin{aligned}(\mathbf{\sinrr} t)(\cdot) &= \begin{pmatrix} t & 0 & 0 \\ 0 & \sinh t & 0 \\ 0 & 0 & \sinh t \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\sinh t & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')\sinh t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\hat{r}\hat{r}'t + \hat{\theta}\hat{\theta}'\sinh t + \hat{z}\hat{z}'\sinh t) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ (\mathbf{\sinrr} t)\hat{r} \times (\cdot) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh t \\ 0 & \sinh t & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')0 & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')0 & (\hat{\theta}\hat{z}')(-\sinh t) \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')\sinh t & (\hat{z}\hat{z}')0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\hat{z}\hat{\theta}' - \hat{\theta}\hat{z}')\sinh t \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}\end{aligned}$$

vector $r\text{-hat}$ double cross basis exponential and Euler's theorem: $\mathbf{exprr}$ and $\mathbf{exprr} = \mathbf{cisrr}$

$$\begin{aligned}\mathbf{exprr} t &= \exp(t \hat{r} \times \hat{r} \times) \\ &= 1 + t(\hat{r} \times \hat{r} \times) + \frac{1}{2!}t^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{3!}t^3(\hat{r} \times \hat{r} \times)^3 + \frac{1}{4!}t^4(\hat{r} \times \hat{r} \times)^4 + \dots \\ &= 1 + t(\hat{r} \times)^2 + \frac{1}{2!}t^2(\hat{r} \times)^4 + \frac{1}{3!}t^3(\hat{r} \times)^6 + \frac{1}{4!}t^4(\hat{r} \times)^8 + \dots \\ &= \left(1 + \frac{1}{2!}t^2 + (\hat{r} \times)^2 + \frac{1}{4!}t^4 + \dots\right) + \left(t\hat{r} \times + \frac{1}{3!}t^3(\hat{r} \times)^3 + \frac{1}{5!}t^5(\hat{r} \times)^5 + \dots\right) \\ &= \left(1 + \frac{1}{2!}t^2(\hat{r} \times)^2 + \frac{1}{4!}t^4(\hat{r} \times)^4 + \dots\right) + \left(t + \frac{1}{3!}t^3(\hat{r} \times)^2 + \frac{1}{5!}t^5(\hat{r} \times)^4 + \dots\right)\hat{r} \times \\ &= \mathbf{cosrr} t + (\mathbf{\sinrr} t)\hat{r} \times \hat{r} \times \\ &= \mathbf{cisrr} t\end{aligned}$$

$\theta\text{-hat}$ direction

$z\text{-hat}$ direction

$r\text{-hat}$ direction

$$\begin{aligned}(\mathbf{exprr} t)\hat{\theta} &= \exp(t \hat{r} \times \hat{r} \times)\hat{\theta} \\ &= \mathbf{cosrr} t \hat{\theta} + (\mathbf{\sinrr} t)\hat{r} \times \hat{r} \times \hat{\theta} \\ &= \mathbf{cosrr} t \hat{\theta} - \mathbf{\sinrr} t \hat{\theta} \\ &= \hat{\theta} \cosh t - \hat{\theta} \sinh t \\ &= \hat{\theta} e^{-t} \\ (\mathbf{exprr} t)\hat{z} &= \exp(t \hat{r} \times \hat{r} \times)\hat{z} \\ &= \mathbf{cosrr} t \hat{z} + (\mathbf{\sinrr} t)\hat{r} \times \hat{r} \times \hat{z} \\ &= \mathbf{cosrr} t \hat{z} - \mathbf{\sinrr} t \hat{z} \\ &= \hat{z} \cosh t - \hat{z} \sinh t \\ &= \hat{z} e^{-t} \\ (\mathbf{exprr} t)\hat{r} &= \exp(t \hat{r} \times \hat{r} \times)\hat{r} \\ &= \mathbf{cosrr} t \hat{r} + (\mathbf{\sinrr} t)\hat{r} \times \hat{r} \times \hat{r} \\ &= \mathbf{cosrr} t \hat{r} + 0 \hat{r} \\ &= \hat{r}\end{aligned}$$

summary of components: $r\text{-hat}$ direction $(\mathbf{exprr} t)\hat{r} = \hat{r}$
 $\theta\text{-hat}$ direction $(\mathbf{exprr} t)\hat{\theta} = \hat{\theta} e^{-t}$
 $z\text{-hat}$ direction $(\mathbf{exprr} t)\hat{z} = \hat{z} e^{-t}$

diadic expansion

$$(\mathbf{exprr} t)(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(-t) & 0 \\ 0 & 0 & \exp(-t) \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')1 & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\exp(-t) & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')\exp(-t) \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \hat{r}\hat{r}' + \hat{\theta}\hat{\theta}'\exp(-t) + \hat{z}\hat{z}'\exp(-t)$$

de-Moivre's theorem (attenuation scaling $\mathbf{cisrr}$)

$$\begin{aligned}(\mathbf{exprr} t)^\omega &= (\exp(t \hat{r} \times \hat{r} \times))^\omega \\ &= \exp(\omega t \hat{r} \times \hat{r} \times) \\ &= 1 + \omega t \hat{r} \times \hat{r} \times + \frac{1}{2!}(\omega t)^2(\hat{r} \times \hat{r} \times)^2 + \frac{1}{3!}(\omega t)^3(\hat{r} \times \hat{r} \times)^3 + \frac{1}{4!}(\omega t)^4(\hat{r} \times \hat{r} \times)^4 + \dots \\ &= \left(1 + \frac{1}{2!}t^2(\hat{r} \times)^4 + \frac{1}{4!}(\omega t)^4(\hat{r} \times)^8 + \dots\right) + \left((\omega t)^2 \hat{r} \times + \frac{1}{3!}(\omega t)^6(\hat{r} \times)^3 + \frac{1}{5!}(\omega t)^5(\hat{r} \times)^{10} + \dots\right) \\ &= \left(1 + \frac{1}{2!}(\omega t)^2(\hat{r} \times)^4 + \frac{1}{4!}(\omega t)^4(\hat{r} \times)^8 + \dots\right) + \left((\omega t)^2 + \frac{1}{3!}(\omega t)^3(\hat{r} \times)^4 + \frac{1}{5!}(\omega t)^5(\hat{r} \times)^8 + \dots\right)\hat{r} \times \\ &= \mathbf{cosr} \omega t + (\mathbf{sinr} \omega t)\hat{r} \times \hat{r} \times \\ &= \mathbf{cisrr} \omega t\end{aligned}$$

vector $\hat{r}$ -hat double cross basis secant

$$(\text{secrr } t)(\cdot) = (\text{cosrr } t)^{-1}(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & 0 \\ 0 & 0 & \cosh t \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{sech } t & 0 \\ 0 & 0 & \text{sech } t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r})1 & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})\text{sech } t & -(\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\text{sech } t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector $\hat{r}$ -hat double cross basis tangent

$$(\text{tanrr } t)\hat{r} \times (\cdot) = (\text{cosrr } t)^{-1}(\text{sinrr } t)\hat{r} \times \hat{r} \times (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{sech } t & 0 \\ 0 & 0 & \text{sech } t \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh t \\ 0 & \sinh t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tanh t \\ 0 & \tanh t & 0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

POLAR CYLINDRICAL DOUBLE CROSS $\hat{\theta} \times \hat{\theta} \times (\cdot)$ BASIS FUNCTIONS: cosqq, sinqq, expqq, secqq, and tanqq

vector $\hat{\theta}$ -hat double cross basis cosine: cosqq

$$\begin{aligned} \text{cosqq } t &= \frac{1}{2} \left(\exp(t \hat{\theta} \times \hat{\theta} \times) + \exp(-t \hat{\theta} \times \hat{\theta} \times) \right) \\ &= \frac{1}{2} \exp(t \hat{\theta} \times \hat{\theta} \times) + \frac{1}{2} \exp(-t \hat{\theta} \times \hat{\theta} \times) \\ &= \frac{1}{2} \left(1 + t(\hat{\theta} \times)^2 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{3!} t^3 (\hat{\theta} \times)^6 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) \\ &\quad + \frac{1}{2} \left(1 - t(\hat{\theta} \times)^2 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 - \frac{1}{3!} t^3 (\hat{\theta} \times)^6 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 - \dots \right) \\ &= 1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \end{aligned}$$

$\hat{r}$ -hat direction

$$\begin{aligned} \text{cosqq } t \hat{r} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) \hat{r} \\ &= \hat{r} + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 \hat{r} + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 \hat{r} + \dots \\ &= \hat{r} + \frac{1}{2!} t^2 \hat{r} + \frac{1}{4!} t^4 \hat{r} + \dots \\ &= \hat{r} \left(1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \dots \right) \\ &= \hat{r} \cosh t \end{aligned}$$

where $(\hat{\theta} \times)^4 \hat{r} = (\hat{\theta} \times)^8 \hat{r} = \hat{r}$

$\hat{z}$ -hat direction

$$\begin{aligned} \text{cosqq } t \hat{z} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) \hat{z} \\ &= \hat{z} + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 \hat{z} + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 \hat{z} + \dots \\ &= \hat{z} + \frac{1}{2!} t^2 \hat{z} + \frac{1}{4!} t^4 \hat{z} + \dots \\ &= \hat{z} \left(1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \dots \right) \\ &= \hat{z} \cosh t \end{aligned}$$

where $(\hat{\theta} \times)^4 \hat{z} = (\hat{\theta} \times)^8 \hat{z} = \hat{z}$

$\hat{\theta}$ -hat direction

$$\begin{aligned} \text{cosqq } t \hat{\theta} &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) \hat{\theta} \\ &= \hat{\theta} + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 \hat{\theta} + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 \hat{\theta} + \dots \\ &= \hat{\theta} + 0 \hat{\theta} + 0 \hat{\theta} + \dots \\ &= \hat{\theta} (1 + 0 + 0 + \dots) \\ &= \hat{\theta} \end{aligned}$$

where $(\hat{\theta} \times)^4 \hat{\theta} = (\hat{\theta} \times)^8 \hat{\theta} = 0 \hat{\theta}$

summary of components: $\hat{r}$ -hat direction **cosqq** $t \hat{r} = \hat{r} \cosh t$

$\hat{\theta}$ -hat direction **cosqq** $t \hat{\theta} = \hat{\theta}$

$\hat{z}$ -hat direction **cosqq** $t \hat{z} = \hat{z} \cosh t$

diadic expansion

$$(\text{cosqq } t)(\cdot) = \begin{pmatrix} \cosh t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r})\cosh t & (\hat{r}\hat{\theta})0 & (\hat{r}\hat{z})0 \\ (\hat{\theta}\hat{r})0 & (\hat{\theta}\hat{\theta})1 & (\hat{\theta}\hat{z})0 \\ (\hat{z}\hat{r})0 & (\hat{z}\hat{\theta})0 & (\hat{z}\hat{z})\cosh t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector $\hat{\theta}$ -hat double cross basis sine: sinqq

$$\begin{aligned} (\text{sinqq } t)\hat{\theta} \times \hat{\theta} \times &= \frac{1}{2} \left(\exp(t \hat{\theta} \times \hat{\theta} \times) - \exp(-t \hat{\theta} \times \hat{\theta} \times) \right) \\ &= \frac{1}{2} \exp(t \hat{\theta} \times \hat{\theta} \times) - \frac{1}{2} \exp(-t \hat{\theta} \times \hat{\theta} \times) \\ &= \frac{1}{2} \left(1 + t(\hat{\theta} \times \hat{\theta} \times) + \frac{1}{2!} t^2 (\hat{\theta} \times \hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times \hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times \hat{\theta} \times)^4 + \dots \right) \\ &\quad - \frac{1}{2} \left(1 - t(\hat{\theta} \times \hat{\theta} \times) + \frac{1}{2!} t^2 (\hat{\theta} \times \hat{\theta} \times)^2 - \frac{1}{3!} t^3 (\hat{\theta} \times \hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times \hat{\theta} \times)^4 - \dots \right) \\ &= t(\hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times)^6 + \frac{1}{5!} t^5 (\hat{\theta} \times)^{10} + \dots \\ \text{sinqq } t &= t + \frac{1}{3!} t^3 (\hat{\theta} \times)^4 + \frac{1}{5!} t^5 (\hat{\theta} \times)^8 + \dots \end{aligned}$$

r -hat direction

$$\begin{aligned} \sin \mathbf{q} \mathbf{q} t \hat{r} &= \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^4 + \frac{1}{5!} t^5 (\hat{\theta} \times)^8 + \dots \right) \hat{r} \\ &= t \hat{r} + \frac{1}{3!} t^3 (\hat{\theta} \times)^4 \hat{r} + \frac{1}{5!} t^5 (\hat{\theta} \times)^8 \hat{r} + \dots \\ &= t \hat{r} + \frac{1}{3!} t^3 \hat{r} + \frac{1}{5!} t^5 \hat{r} + \dots \\ &= \hat{r} \left(t + \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \dots \right) \\ &= \hat{r} \sinh t \end{aligned}$$

$$\text{where } (\hat{\theta} \times)^4 \hat{r} = (\hat{\theta} \times)^8 \hat{r} = \hat{r}$$

summary of components: r -hat direction $\sin \mathbf{q} \mathbf{q} t \hat{r} = \hat{r} \sinh t$

θ -hat direction $\sin \mathbf{q} \mathbf{q} t \hat{\theta} = \hat{\theta} t$

z -hat direction $\sin \mathbf{q} \mathbf{q} t \hat{z} = \hat{z} \sinh t$

diadic expansion

$$(\sin \mathbf{q} \mathbf{q} t)(\cdot) = \begin{pmatrix} \sinh t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & \sinh t \end{pmatrix} \underset{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r}') \sinh t & (\hat{r}\hat{\theta}') 0 & (\hat{r}\hat{z}') 0 \\ (\hat{\theta}\hat{r}') 0 & (\hat{\theta}\hat{\theta}') t & (\hat{\theta}\hat{z}') 0 \\ (\hat{z}\hat{r}') 0 & (\hat{z}\hat{\theta}') 0 & (\hat{z}\hat{z}') \sinh t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$(\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times (\cdot) = \begin{pmatrix} 0 & 0 & \sinh t \\ 0 & 0 & 0 \\ -\sinh t & 0 & 0 \end{pmatrix} \underset{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r}') 0 & (\hat{r}\hat{\theta}') 0 & (\hat{r}\hat{z}') \sinh t \\ (\hat{\theta}\hat{r}') 0 & (\hat{\theta}\hat{\theta}') 0 & (\hat{\theta}\hat{z}') 0 \\ (\hat{z}\hat{r}') (-\sinh t) & (\hat{z}\hat{\theta}') 0 & (\hat{z}\hat{z}') 0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector θ -hat double cross basis exponential and Euler's theorem: $\exp \mathbf{q} \mathbf{q}$ and $\exp \mathbf{q} \mathbf{q} = \text{cis} \mathbf{q} \mathbf{q}$

$$\begin{aligned} \exp \mathbf{q} \mathbf{q} t &= \exp(t \hat{\theta} \times \hat{\theta} \times) \\ &= 1 + t(\hat{\theta} \times \hat{\theta} \times) + \frac{1}{2!} t^2 (\hat{\theta} \times \hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times \hat{\theta} \times)^3 + \frac{1}{4!} t^4 (\hat{\theta} \times \hat{\theta} \times)^4 + \dots \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) + \left(t(\hat{\theta} \times)^2 + \frac{1}{3!} t^3 (\hat{\theta} \times)^6 + \frac{1}{5!} t^5 (\hat{\theta} \times)^{10} + \dots \right) \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} t^4 (\hat{\theta} \times)^8 + \dots \right) + \left(t + \frac{1}{3!} t^3 (\hat{\theta} \times)^4 + \frac{1}{5!} t^5 (\hat{\theta} \times)^8 + \dots \right) \hat{\theta} \times \hat{\theta} \times \\ &= \cos \mathbf{q} \mathbf{q} t + (\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times \\ &= \text{cis} \mathbf{q} \mathbf{q} t \end{aligned}$$

r -hat direction

$$\begin{aligned} (\exp \mathbf{q} \mathbf{q} t) \hat{r} &= \exp(t \hat{\theta} \times \hat{\theta} \times) \hat{r} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{r} + (\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times \hat{r} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{r} - \sin \mathbf{q} \mathbf{q} t \hat{r} \\ &= \hat{r} \cosh t - \hat{r} \sinh t \\ &= \hat{r} e^{-t} \end{aligned}$$

z -hat direction

$$\begin{aligned} (\exp \mathbf{q} \mathbf{q} t) \hat{z} &= \exp(t \hat{\theta} \times \hat{\theta} \times) \hat{z} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{z} + (\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times \hat{z} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{z} - \sin \mathbf{q} \mathbf{q} t \hat{z} \\ &= \hat{z} \cos t - \hat{z} \sin t \\ &= \hat{z} e^{-t} \end{aligned}$$

θ -hat direction

$$\begin{aligned} (\exp \mathbf{q} \mathbf{q} t) \hat{\theta} &= \exp(t \hat{\theta} \times \hat{\theta} \times) \hat{\theta} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{\theta} + (\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times \hat{\theta} \\ &= \cos \mathbf{q} \mathbf{q} t \hat{\theta} + 0 \hat{\theta} \\ &= \hat{\theta} \end{aligned}$$

summary of components: r -hat direction $(\exp \mathbf{q} \mathbf{q} t) \hat{r} = \hat{r} \exp(-t)$

θ -hat direction $(\exp \mathbf{q} \mathbf{q} t) \hat{\theta} = \hat{\theta}$

z -hat direction $(\exp \mathbf{q} \mathbf{q} t) \hat{z} = \hat{z} \exp(-t)$

diadic expansion

$$(\exp \mathbf{q} \mathbf{q} t)(\cdot) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \underset{\text{polar basis}}{=} \begin{pmatrix} (\hat{r}\hat{r}') e^{-t} & (\hat{r}\hat{\theta}') 0 & (\hat{r}\hat{z}') 0 \\ (\hat{\theta}\hat{r}') 0 & (\hat{\theta}\hat{\theta}') 1 & -(\hat{\theta}\hat{z}') 0 \\ (\hat{z}\hat{r}') 0 & (\hat{z}\hat{\theta}') 0 & (\hat{z}\hat{z}') e^{-t} \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

de-Moivre's theorem (attenuation scaling cisqq)

$$\begin{aligned}
 (\exp \mathbf{q} \mathbf{q} t)^{\omega} &= \left(\exp(t \hat{\theta} \times \hat{\theta} \times) \right)^{\omega} \\
 &= \exp(\omega t \hat{\theta} \times \hat{\theta} \times) \\
 &= 1 + \omega t (\hat{\theta} \times \hat{\theta} \times) + \frac{1}{2!} (\omega t)^2 (\hat{\theta} \times \hat{\theta} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{\theta} \times \hat{\theta} \times)^3 + \frac{1}{4!} (\omega t)^4 (\hat{\theta} \times \hat{\theta} \times)^4 + \dots \\
 &= \left(1 + \frac{1}{2!} t^2 (\hat{\theta} \times)^4 + \frac{1}{4!} (\omega t)^4 (\hat{\theta} \times)^8 + \dots \right) + \left((\omega t) (\hat{\theta} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{\theta} \times)^6 + \frac{1}{5!} (\omega t)^5 (\hat{\theta} \times)^{10} + \dots \right) \\
 &= \left(1 + \frac{1}{2!} (\omega t)^2 (\hat{\theta} \times)^4 + \frac{1}{4!} (\omega t)^4 (\hat{\theta} \times)^8 + \dots \right) + \left(\omega t + \frac{1}{3!} (\omega t)^3 (\hat{\theta} \times)^4 + \frac{1}{5!} (\omega t)^5 (\hat{\theta} \times)^8 + \dots \right) \hat{\theta} \times \hat{\theta} \times \\
 &= \cos \mathbf{q} \mathbf{q} \omega t + (\sin \mathbf{q} \mathbf{q} \omega t) \hat{\theta} \times \hat{\theta} \times \\
 &= \text{cis} \mathbf{q} \mathbf{q} \omega t
 \end{aligned}$$

vector θ -hat double cross basis secant

$$(\text{sec} \mathbf{q} \mathbf{q})(\cdot) = (\cos \mathbf{q} \mathbf{q} t)^{-1}(\cdot) = \begin{pmatrix} \cosh t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh t \end{pmatrix}^{-1} = \begin{pmatrix} \text{sech } t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{sech } t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r}') \text{sech } t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')1 & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}') \text{sech } t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector θ -hat double cross basis tangent

$$(\tan \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times (\cdot) = (\cos \mathbf{q} \mathbf{q} t)^{-1} (\sin \mathbf{q} \mathbf{q} t) \hat{\theta} \times \hat{\theta} \times (\cdot) = \begin{pmatrix} \text{sech } t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{sech } t \end{pmatrix} \begin{pmatrix} 0 & 0 & \sinh t \\ 0 & 0 & 0 \\ -\sinh t & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tanh t \\ 0 & 0 & 0 \\ -\tanh t & 0 & 0 \end{pmatrix}$$

POLAR CYLINDRICAL DOUBLE CROSS $\hat{z} \times \hat{z} \times (\cdot)$ BASIS FUNCTIONS: $\cos \mathbf{z} \mathbf{z}$, $\sin \mathbf{z} \mathbf{z}$, $\exp \mathbf{z} \mathbf{z}$, $\sec \mathbf{z} \mathbf{z}$, and $\tan \mathbf{z} \mathbf{z}$

vector z -hat double cross basis cosine: $\cos \mathbf{z} \mathbf{z}$

$$\begin{aligned}
 \cos \mathbf{z} \mathbf{z} t &= \frac{1}{2} (\exp(t \hat{z} \times \hat{z} \times) + \exp(-t \hat{z} \times \hat{z} \times)) \\
 &= \frac{1}{2} \exp(t \hat{z} \times \hat{z} \times) + \frac{1}{2} \exp(-t \hat{z} \times \hat{z} \times) \\
 &= \frac{1}{2} \left(1 + t \hat{z} \times \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 + \dots \right) \\
 &\quad + \frac{1}{2} \left(1 - t \hat{z} \times \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 - \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 - \dots \right) \\
 &= 1 + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 + \dots \\
 &= 1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots
 \end{aligned}$$

r -hat direction

θ -hat direction

z -hat direction

$ \begin{aligned} \cos \mathbf{z} \mathbf{z} t \hat{r} &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \right) \hat{r} \\ &= \hat{r} + \frac{1}{2!} t^2 (\hat{z} \times)^4 \hat{r} + \frac{1}{4!} t^4 (\hat{z} \times)^8 \hat{r} + \dots \\ &= \hat{r} + \frac{1}{2!} t^2 \hat{r} + \frac{1}{4!} t^4 \hat{r} + \dots \\ &= \hat{r} \left(1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \dots \right) \\ &= \hat{r} \cosh t \end{aligned} $	$ \begin{aligned} \cos \mathbf{z} \mathbf{z} t \hat{\theta} &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \right) \hat{\theta} \\ &= \hat{\theta} + \frac{1}{2!} t^2 (\hat{z} \times)^4 \hat{\theta} + \frac{1}{4!} t^4 (\hat{z} \times)^8 \hat{\theta} + \dots \\ &= \hat{\theta} + \frac{1}{2!} t^2 \hat{\theta} + \frac{1}{4!} t^4 \hat{\theta} + \dots \\ &= \hat{\theta} \left(1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \dots \right) \\ &= \hat{\theta} \cosh t \end{aligned} $	$ \begin{aligned} \cos \mathbf{z} \mathbf{z} t \hat{z} &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \right) \hat{z} \\ &= \hat{z} + \frac{1}{2!} t^2 (\hat{z} \times)^4 \hat{z} + \frac{1}{4!} t^4 (\hat{z} \times)^8 \hat{z} + \dots \\ &= \hat{z} + 0 \hat{z} + 0 \hat{z} + \dots \\ &= \hat{z} (1 + 0 + 0 + \dots) \\ &= \hat{z} \end{aligned} $
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where $(\hat{z} \times)^4 \hat{r} = (\hat{z} \times)^8 \hat{r} = \hat{r}$

where $(\hat{z} \times)^4 \hat{\theta} = (\hat{z} \times)^8 \hat{\theta} = \hat{\theta}$

where $(\hat{z} \times)^4 \hat{z} = (\hat{z} \times)^8 \hat{z} = 0 \hat{z}$

summary of components: r -hat direction $\sin \mathbf{z} \mathbf{z} t \hat{r} = \hat{r} \sinh t$
 θ -hat direction $\sin \mathbf{z} \mathbf{z} t \hat{\theta} = \hat{\theta} \sinh t$
 z -hat direction $\sin \mathbf{z} \mathbf{z} t \hat{z} = \hat{z} t$

diadic expansion

$$(\cos \mathbf{z} \mathbf{z} t)(\cdot) = \begin{pmatrix} \cosh t & 0 & 0 \\ 0 & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r}') \cosh t & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}') \cosh t & (\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector z-hat double cross basis sine: sinzz

$$\begin{aligned}
 (\sinzz t) \hat{z} \times \hat{z} \times &= \frac{1}{2} (\exp(t \hat{z} \times \hat{z} \times) - \exp(-t \hat{z} \times \hat{z} \times)) \\
 &= \frac{1}{2} \exp(t \hat{z} \times \hat{z} \times) - \frac{1}{2} \exp(-t \hat{z} \times \hat{z} \times) \\
 &= \frac{1}{2} \left(1 + t \hat{z} \times \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 + \dots \right) \\
 &\quad - \frac{1}{2} \left(1 - t \hat{z} \times \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 - \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 - \dots \right) \\
 &= t \hat{z} \times \hat{z} \times + \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{5!} t^5 (\hat{z} \times \hat{z} \times)^5 + \dots \\
 &= t (\hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^6 + \dots \\
 \sinzz t &= t + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^8 + \dots
 \end{aligned}$$

r-hat direction

$$\begin{aligned}
 \sinzz t \hat{r} &= \left(t + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^8 + \dots \right) \hat{r} \\
 &= t \hat{r} + \frac{1}{3!} t^3 (\hat{z} \times)^4 \hat{r} + \frac{1}{5!} t^5 (\hat{z} \times)^8 \hat{r} + \dots \\
 &= t \hat{r} + \frac{1}{3!} t^3 \hat{r} + \frac{1}{5!} t^5 \hat{r} + \dots \\
 &= \hat{r} \left(t + \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \dots \right) \\
 &= \hat{r} \sinh t
 \end{aligned}$$

$$\text{where } (\hat{z} \times)^4 \hat{r} = (\hat{z} \times)^8 \hat{r} = \hat{r}$$

θ -hat direction

$$\begin{aligned}
 \sinzz t \hat{\theta} &= \left(t + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^8 + \dots \right) \hat{\theta} \\
 &= t \hat{\theta} + \frac{1}{3!} t^3 (\hat{z} \times)^4 \hat{\theta} + \frac{1}{5!} t^5 (\hat{z} \times)^8 \hat{\theta} + \dots \\
 &= t \hat{\theta} + \frac{1}{3!} t^3 \hat{\theta} + \frac{1}{5!} t^5 \hat{\theta} + \dots \\
 &= \hat{\theta} \left(t + \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \dots \right) \\
 &= \hat{\theta} \sinh t
 \end{aligned}$$

$$\text{where } (\hat{\theta} \times)^4 \hat{r} = (\hat{\theta} \times)^8 \hat{r} = \hat{r}$$

z-hat direction

$$\begin{aligned}
 \sinzz t \hat{z} &= \left(t + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^8 + \dots \right) \hat{z} \\
 &= t \hat{z} + \frac{1}{3!} t^3 (\hat{z} \times)^4 \hat{z} + \frac{1}{5!} t^5 (\hat{z} \times)^8 \hat{z} + \dots \\
 &= t \hat{z} + 0 \hat{z} + 0 \hat{z} + \dots \\
 &= \hat{z} (t + 0 + 0 + \dots) \\
 &= \hat{z} t
 \end{aligned}$$

$$\text{where } (\hat{z} \times)^4 \hat{z} = (\hat{z} \times)^8 \hat{z} = 0 \hat{z}$$

summary of components: *r*-hat direction **coszz** $t \hat{r} = \hat{r} \cosh t$
 θ -hat direction **coszz** $t \hat{\theta} = \hat{\theta} \cosh t$
z-hat direction **coszz** $t \hat{z} = \hat{z}$

diadic expansion

$$\begin{aligned}
 (\sinzz t) (\cdot) &= \begin{pmatrix} \sinh t & 0 & 0 \\ 0 & \sinh t & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r}) \sinh t & (\hat{r}\hat{\theta}) 0 & (\hat{r}\hat{z}) 0 \\ (\hat{\theta}\hat{r}) 0 & (\hat{\theta}\hat{\theta}) \sinh t & (\hat{\theta}\hat{z}) 0 \\ (\hat{z}\hat{r}) 0 & (\hat{z}\hat{\theta}) 0 & (\hat{z}\hat{z}) t \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \\
 (\sinzz t) \hat{r} \times \hat{r} \times (\cdot) &= \begin{pmatrix} 0 & -\sinh t & 0 \\ \sinh t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (\hat{r}\hat{r}) 0 & (\hat{r}\hat{\theta}) (-\sinh t) & (\hat{r}\hat{z}) 0 \\ (\hat{\theta}\hat{r}) \sinh t & (\hat{\theta}\hat{\theta}) 0 & (\hat{\theta}\hat{z}) 0 \\ (\hat{z}\hat{r}) 0 & (\hat{z}\hat{\theta}) 0 & (\hat{z}\hat{z}) 0 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}
 \end{aligned}$$

vector z-hat double cross basis exponential and Euler's theorem: expzz and expzz = ciszz

$$\begin{aligned}
 \expzz t &= \exp(t \hat{z} \times \hat{z} \times) \\
 &= 1 + t \hat{z} \times \hat{z} \times + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} t^4 (\hat{z} \times \hat{z} \times)^4 + \dots \\
 &= 1 + t (\hat{z} \times)^2 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{3!} t^3 (\hat{z} \times)^6 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \\
 &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \right) + \left(t (\hat{z} \times)^2 + \frac{1}{3!} t^3 (\hat{z} \times)^6 + \frac{1}{5!} t^5 (\hat{z} \times)^{10} + \dots \right) \\
 &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times)^4 + \frac{1}{4!} t^4 (\hat{z} \times)^8 + \dots \right) + \left(t + \frac{1}{3!} t^3 (\hat{z} \times)^4 + \frac{1}{5!} t^5 (\hat{z} \times)^8 + \dots \right) \hat{z} \times \hat{z} \times \\
 &= \mathbf{coszz} t + (\sinzz t) \hat{z} \times \hat{z} \times \\
 &= \mathbf{ciszz} t
 \end{aligned}$$

r-hat direction

$$\begin{aligned}
 (\expzz t) \hat{r} &= \exp(t \hat{z} \times \hat{z} \times) \hat{r} \\
 &= \mathbf{coszz} t \hat{r} + (\sinzz t) \hat{z} \times \hat{z} \times \hat{r} \\
 &= \mathbf{coszz} t \hat{r} - \sinzz t \hat{r} \\
 &= \hat{r} \cosh t - \hat{r} \sinh t \\
 &= \hat{r} e^{-t}
 \end{aligned}$$

θ -hat direction

$$\begin{aligned}
 (\expzz t) \hat{\theta} &= \exp(t \hat{z} \times \hat{z} \times) \hat{\theta} \\
 &= \mathbf{coszz} t \hat{\theta} + (\sinzz t) \hat{r} \times \hat{r} \times \hat{\theta} \\
 &= \mathbf{coszz} t \hat{\theta} - \sinzz t \hat{\theta} \\
 &= \hat{\theta} \cosh t - \hat{\theta} \sinh t \\
 &= \hat{\theta} e^{-t}
 \end{aligned}$$

z-hat direction

$$\begin{aligned}
 (\expzz t) \hat{z} &= \exp(t \hat{z} \times \hat{z} \times) \hat{z} \\
 &= \mathbf{coszz} t \hat{z} + (\sinzz t) \hat{z} \times \hat{z} \times \hat{z} \\
 &= \mathbf{coszz} t \hat{z} + 0 \hat{z} \\
 &= \hat{z}
 \end{aligned}$$

summary of components: *r*-hat direction **(expzz)** $t \hat{r} = \hat{r} e^{-t}$

$$\begin{aligned}\theta\text{-hat direction } (\mathbf{expzz} t) \hat{\theta} &= \hat{\theta} e^{-t} \\ z\text{-hat direction } (\mathbf{expzz} t) \hat{z} &= \hat{z}\end{aligned}$$

diadic expansion

$$(\mathbf{expzz} t)(\cdot) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')e^{-t} & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')e^{-t} & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

de-Moivre's theorem (attenuation scaling ciszz)

$$\begin{aligned}(\mathbf{expzz} t)^\omega &= (\exp(t \hat{z} \times \hat{z} \times))^\omega \\ &= \exp(\omega t \hat{z} \times \hat{z} \times) \\ &= 1 + \omega t \hat{z} \times \hat{z} \times + \frac{1}{2!} (\omega t)^2 (\hat{z} \times \hat{z} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{z} \times \hat{z} \times)^3 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times \hat{z} \times)^4 + \dots \\ &= 1 + \omega t (\hat{z} \times)^2 + \frac{1}{2!} (\omega t)^2 (\hat{z} \times)^4 + \frac{1}{3!} (\omega t)^3 (\hat{z} \times)^6 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times)^8 + \dots \\ &= \left(1 + \frac{1}{2!} t^2 (\hat{z} \times \hat{z} \times)^4 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times \hat{z} \times)^8 + \dots\right) + \left((\omega t) (\hat{z} \times \hat{z} \times)^2 + \frac{1}{3!} (\omega t)^3 (\hat{z} \times \hat{z} \times)^6 + \frac{1}{5!} (\omega t)^5 (\hat{z} \times \hat{z} \times)^{10} + \dots\right) \\ &= \left(1 + \frac{1}{2!} (\omega t)^2 (\hat{z} \times)^4 + \frac{1}{4!} (\omega t)^4 (\hat{z} \times)^8 + \dots\right) + \left(\omega t + \frac{1}{3!} (\omega t)^3 (\hat{z} \times)^4 + \frac{1}{5!} (\omega t)^5 (\hat{z} \times)^8 + \dots\right) \hat{z} \times \hat{z} \times \\ &= \mathbf{coszz} \omega t + (\mathbf{sinzz} \omega t) \hat{z} \times \hat{z} \times \\ &= \mathbf{ciszz} \omega t\end{aligned}$$

vector z-hat double cross basis secant

$$(\mathbf{seczz} t)(\cdot) = (\mathbf{coszz} t)^{-1}(\cdot) = \begin{pmatrix} \cosh t & 0 & 0 \\ 0 & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}}^{-1} = \begin{pmatrix} \sec ht & 0 & 0 \\ 0 & \sec ht & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} (\hat{r}\hat{r}')\sec ht & (\hat{r}\hat{\theta}')0 & (\hat{r}\hat{z}')0 \\ (\hat{\theta}\hat{r}')0 & (\hat{\theta}\hat{\theta}')\sec ht & -(\hat{\theta}\hat{z}')0 \\ (\hat{z}\hat{r}')0 & (\hat{z}\hat{\theta}')0 & (\hat{z}\hat{z}')1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

vector z-hat double cross basis tangent

$$(\mathbf{tanzz} t) \hat{z} \times \hat{z} \times (\cdot) = (\mathbf{coszz} t)^{-1} (\mathbf{sinzz} t) \hat{z} \times \hat{z} \times (\cdot) = \begin{pmatrix} \operatorname{sech} t & 0 & 0 \\ 0 & \operatorname{sech} t & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\text{polar basis}} \begin{pmatrix} 0 & -\sinh t & 0 \\ \sinh t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{polar basis}} = \begin{pmatrix} 0 & -\tanh t & 0 \\ \tanh t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{polar basis}}$$

SECTION 3 – EXPANSION OF POLAR BASIS FUNCTIONS IN STANDARD BASIS

Writing sin and cos in equations, mathematical working, and matrices ends up being a substantial amount of writing, especially when things get complicated and especially for matrices. So as an additional shorthand I often write $\sin \theta = s_\theta$ and $\cos \theta = c_\theta$ for the scalar sine and cosine which I do within this topic for expanding out the polar matrix basis functions in terms of the standard rectangular basis. These matrix expressions would be almost too unwieldy otherwise. The other way to overcome this difficulty is to work with the change of basis matrices which keeps things concise when working with a programmable calculator. The 3 bases relevant to this section are

(1) Cylindrical polar row vector basis (2) Cylindrical polar column vector basis (3) Cylindrical polar matrix basis

A vector or a matrix where the vector or matrix base is unspecified, that whereby the basis components are unshown, is **rectangular by default** unless there is a subscript stating **polar basis**.

For z-axis (**expk**) cylindricals [as apposed to the less common x-axis or y-axis cylindricals (**expi** and **expj**)], the basis components of the **polar cylindrical basis-exponential** are

$$\hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0 \quad \hat{\theta} = -\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0 \quad \text{and} \quad \hat{z} = \hat{i} 0 + \hat{j} 0 + \hat{k} 1$$

that is, the **polar column vector basis components** expressed in terms of the **standard rectangular column bases** are

$$\hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} = (\cos \theta \quad \sin \theta \quad 0)', \quad \hat{\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} = (-\sin \theta \quad \cos \theta \quad 0)', \text{ and} \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0 \quad 0 \quad 1)'$$

and the **polar row vector basis components** expressed in terms of the **standard rectangular row bases** are

$$\hat{r}' = (\cos \theta \quad \sin \theta \quad 0) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}', \quad \hat{\theta}' = (-\sin \theta \quad \cos \theta \quad 0) = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}', \text{ and} \quad \hat{z}' = (0 \quad 0 \quad 1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}'$$

The **polar vector bases**, expressed in terms of the **standard rectangular basis** with the **standard rectangular components shown explicitly** are

$$\hat{r} = \begin{pmatrix} c_\theta \hat{i} \\ s_\theta \hat{j} \\ 0 \hat{k} \end{pmatrix} \quad \hat{\theta} = \begin{pmatrix} -s_\theta \hat{i} \\ c_\theta \hat{j} \\ 0 \hat{k} \end{pmatrix} \quad \hat{z} = \begin{pmatrix} 0 \hat{i} \\ 0 \hat{j} \\ 1 \hat{k} \end{pmatrix} \quad \hat{r}' = (c_\theta \hat{i}' \quad s_\theta \hat{j}' \quad 0 \hat{k}') \quad \hat{\theta}' = (-s_\theta \hat{i}' \quad c_\theta \hat{j}' \quad 0 \hat{k}') \quad \hat{z}' = (0 \hat{i}' \quad 0 \hat{j}' \quad 1 \hat{k}')$$

The **polar cylindrical matrix basis components** are then the tensor order expansion (outer product) to the matrix basis. The cross-product, $\times$, as a binary mathematical operator is implied but unshown as I will be discussing in the document I have titled *the almighty cross product* (changed from *the generalized cross product*). The polar cylindrical matrix basis components are

$$\hat{r} \hat{r}', \hat{r} \hat{\theta}', \hat{r} \hat{z}', \hat{\theta} \hat{r}', \hat{\theta} \hat{\theta}', \hat{\theta} \hat{z}', \hat{z} \hat{r}', \hat{z} \hat{\theta}', \text{ and } \hat{z} \hat{z}' \quad \text{from} \quad \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix} \begin{pmatrix} \hat{r}' & \hat{\theta}' & \hat{z}' \end{pmatrix}$$

which outer expands [which I contend in the *almighty cross product* that this is also a cross product] to

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{pmatrix} \begin{pmatrix} \hat{r}' & \hat{\theta}' & \hat{z}' \end{pmatrix} = \begin{pmatrix} \alpha_{11} \hat{r} \hat{r}' & \alpha_{12} \hat{r} \hat{\theta}' & \alpha_{13} \hat{r} \hat{z}' \\ \alpha_{21} \hat{\theta} \hat{r}' & \alpha_{22} \hat{\theta} \hat{\theta}' & \alpha_{23} \hat{\theta} \hat{z}' \\ \alpha_{31} \hat{z} \hat{r}' & \alpha_{32} \hat{z} \hat{\theta}' & \alpha_{33} \hat{z} \hat{z}' \end{pmatrix} = \alpha_{11} \hat{r} \hat{r}' + \alpha_{12} \hat{r} \hat{\theta}' + \alpha_{13} \hat{r} \hat{z}' + \alpha_{21} \hat{\theta} \hat{r}' + \alpha_{22} \hat{\theta} \hat{\theta}' + \alpha_{23} \hat{\theta} \hat{z}' + \alpha_{31} \hat{z} \hat{r}' + \alpha_{32} \hat{z} \hat{\theta}' + \alpha_{33} \hat{z} \hat{z}'$$

The **polar cylindrical matrix basis components** can be expressed in terms of the standard **rectangular basis** whereby, as usual, with the standard rectangular basis being silent/latent but generating the matrix format. This shows the details of the polar matrix basis. These are

$$\hat{r} \hat{r}' = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{r} \hat{\theta}' = \begin{pmatrix} -\cos \theta \sin \theta & \cos^2 \theta & 0 \\ -\sin^2 \theta & \sin \theta \cos \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{r} \hat{z}' = \begin{pmatrix} 0 & 0 & \cos \theta \\ 0 & 0 & \sin \theta \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\theta} \hat{r}' = \begin{pmatrix} -\sin \theta \cos \theta & -\sin^2 \theta & 0 \\ \cos^2 \theta & \cos \theta \sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{\theta} \hat{\theta}' = \begin{pmatrix} \sin^2 \theta & -\sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta & \cos^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{\theta} \hat{z}' = \begin{pmatrix} 0 & 0 & -\sin \theta \\ 0 & 0 & \cos \theta \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{z} \hat{r}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos \theta & \sin \theta & 0 \end{pmatrix} \quad \hat{z} \hat{\theta}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 \end{pmatrix} \quad \hat{z} \hat{z}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or in shorthand for the scalar sines and cosines

$$\begin{aligned} \hat{r} \hat{r}' &= \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{r} \hat{\theta}' &= \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{r} \hat{z}' &= \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{\theta} \hat{r}' &= \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{\theta} \hat{\theta}' &= \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{\theta} \hat{z}' &= \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{z} \hat{r}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} & \hat{z} \hat{\theta}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} & \hat{z} \hat{z}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Showing the $3 \times 3 = 9$ **polar matrix basis components** in terms of the **underlying standard rectangular basis**. This makes it clear what basis is what, rather than working in the dark, and also shows that the **cylindrical basis system** is **scalings of the rectangular system** and still further shows which scalings are **associated with rows (primed)** and which are **associated with columns (unprimed)**. This render matrix algebra a system of complex-number like manipulations with the crosses (on i -hat, j -hat, and k -hat) of the cross products silent (might leave the hats silent as well). I discuss this with perhaps too much detail in the **almighty cross product**.

This shows that there is value in keeping the outer product terms coupled with the basis components, with the basis components shown explicitly. I am getting the strong suggestion that there will be very important instances where the components of the vector bases in the outer product expansion to a diadic derived matrix basis must be kept coupled with their vector bases [with colour vectors for example]]. This may prove to be very important mathematics, especially if triadic systems get worked out. It certainly does get cluttered though.

$$\begin{aligned} \hat{r} \hat{r}' &= \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_\theta^2 \hat{i}\hat{i}' & c_\theta s_\theta \hat{i}\hat{j}' & 0 \hat{i}\hat{k}' \\ s_\theta c_\theta \hat{j}\hat{i}' & s_\theta^2 \hat{j}\hat{j}' & 0 \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (c_\theta \hat{i})(c_\theta \hat{i}') & (c_\theta \hat{i})(s_\theta \hat{j}') & (c_\theta \hat{i})(0 \hat{k}') \\ (s_\theta \hat{j})(c_\theta \hat{i}') & (s_\theta \hat{j})(s_\theta \hat{j}') & (s_\theta \hat{j})(0 \hat{k}') \\ (0 \hat{k})(c_\theta \hat{i}') & (0 \hat{k})(s_\theta \hat{j}') & (0 \hat{k})(0 \hat{k}') \end{pmatrix} \\ \hat{r} \hat{\theta}' &= \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -c_\theta s_\theta \hat{i}\hat{i}' & c_\theta^2 \hat{i}\hat{j}' & 0 \hat{i}\hat{k}' \\ -s_\theta^2 \hat{j}\hat{i}' & s_\theta c_\theta \hat{j}\hat{j}' & 0 \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (c_\theta \hat{i})(-s_\theta \hat{i}') & (c_\theta \hat{i})(c_\theta \hat{j}') & (c_\theta \hat{i})(0 \hat{k}') \\ (s_\theta \hat{j})(-s_\theta \hat{i}') & (s_\theta \hat{j})(c_\theta \hat{j}') & (s_\theta \hat{j})(0 \hat{k}') \\ (0 \hat{k})(-s_\theta \hat{i}') & (0 \hat{k})(c_\theta \hat{j}') & (0 \hat{k})(0 \hat{k}') \end{pmatrix} \\ \hat{r} \hat{z}' &= \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \hat{i}\hat{i}' & 0 \hat{i}\hat{j}' & c_\theta \hat{i}\hat{k}' \\ 0 \hat{j}\hat{i}' & 0 \hat{j}\hat{j}' & s_\theta \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (c_\theta \hat{i})(c_\theta \hat{i}') & (c_\theta \hat{i})(0 \hat{j}') & (c_\theta \hat{i})(1 \hat{k}') \\ (s_\theta \hat{j})(c_\theta \hat{i}') & (s_\theta \hat{j})(0 \hat{j}') & (s_\theta \hat{j})(1 \hat{k}') \\ (0 \hat{k})(c_\theta \hat{i}') & (0 \hat{k})(0 \hat{j}') & (0 \hat{k})(1 \hat{k}') \end{pmatrix} \\ \hat{\theta} \hat{r}' &= \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -s_\theta c_\theta \hat{i}\hat{i}' & -s_\theta^2 \hat{i}\hat{j}' & 0 \hat{i}\hat{k}' \\ c_\theta^2 \hat{j}\hat{i}' & c_\theta s_\theta \hat{j}\hat{j}' & 0 \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (-s_\theta \hat{i})(c_\theta \hat{i}') & (-s_\theta \hat{i})(s_\theta \hat{j}') & (-s_\theta \hat{i})(0 \hat{k}') \\ (c_\theta \hat{j})(c_\theta \hat{i}') & (c_\theta \hat{j})(s_\theta \hat{j}') & (c_\theta \hat{j})(0 \hat{k}') \\ (0 \hat{k})(c_\theta \hat{i}') & (0 \hat{k})(s_\theta \hat{j}') & (0 \hat{k})(0 \hat{k}') \end{pmatrix} \\ \hat{\theta} \hat{\theta}' &= \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -s_\theta^2 \hat{i}\hat{i}' & -s_\theta c_\theta \hat{i}\hat{j}' & 0 \hat{i}\hat{k}' \\ -s_\theta c_\theta \hat{j}\hat{i}' & c_\theta^2 \hat{j}\hat{j}' & 0 \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (-s_\theta \hat{i})(-s_\theta \hat{i}') & (-s_\theta \hat{i})(-s_\theta \hat{j}') & (-s_\theta \hat{i})(-s_\theta \hat{k}') \\ (c_\theta \hat{j})(c_\theta \hat{i}') & (c_\theta \hat{j})(c_\theta \hat{j}') & (c_\theta \hat{j})(c_\theta \hat{k}') \\ (0 \hat{k})(0 \hat{i}') & (0 \hat{k})(0 \hat{j}') & (0 \hat{k})(0 \hat{k}') \end{pmatrix} \\ \hat{\theta} \hat{z}' &= \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \hat{i}\hat{i}' & 0 \hat{i}\hat{j}' & -s_\theta \hat{i}\hat{k}' \\ 0 \hat{j}\hat{i}' & 0 \hat{j}\hat{j}' & c_\theta \hat{j}\hat{k}' \\ 0 \hat{k}\hat{i}' & 0 \hat{k}\hat{j}' & 0 \hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (-s_\theta \hat{i})(0 \hat{i}') & (-s_\theta \hat{i})(0 \hat{j}') & (-s_\theta \hat{i})(1 \hat{k}') \\ (c_\theta \hat{j})(0 \hat{i}') & (c_\theta \hat{j})(0 \hat{j}') & (c_\theta \hat{j})(1 \hat{k}') \\ (0 \hat{k})(0 \hat{i}') & (0 \hat{k})(0 \hat{j}') & (0 \hat{k})(1 \hat{k}') \end{pmatrix} \end{aligned}$$

$$\begin{aligned}\hat{z}\hat{r}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ c_\theta\hat{k}\hat{i}' & s_\theta\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (0\hat{i})(c_\theta\hat{i}') & (0\hat{i})(s_\theta\hat{j}') & (0\hat{i})(0\hat{k}') \\ (0\hat{j})(c_\theta\hat{i}') & (0\hat{j})(s_\theta\hat{j}') & (0\hat{j})(0\hat{k}') \\ (1\hat{k})(c_\theta\hat{i}') & (1\hat{k})(s_\theta\hat{j}') & (1\hat{k})(0\hat{k}') \end{pmatrix} \\ \hat{z}\hat{\theta}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ -s_\theta\hat{k}\hat{i}' & c_\theta\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (0\hat{i})(-s_\theta\hat{i}') & (0\hat{i})(c_\theta\hat{j}') & (0\hat{i})(0\hat{k}') \\ (0\hat{j})(-s_\theta\hat{i}') & (0\hat{j})(c_\theta\hat{j}') & (0\hat{j})(0\hat{k}') \\ (1\hat{k})(-s_\theta\hat{i}') & (1\hat{k})(c_\theta\hat{j}') & (1\hat{k})(0\hat{k}') \end{pmatrix} \\ \hat{z}\hat{z}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 1\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} (0\hat{i})(0\hat{i}') & (0\hat{i})(0\hat{j}') & (0\hat{i})(1\hat{k}') \\ (0\hat{j})(0\hat{i}') & (0\hat{j})(0\hat{j}') & (0\hat{j})(1\hat{k}') \\ (1\hat{k})(0\hat{i}') & (1\hat{k})(0\hat{j}') & (1\hat{k})(1\hat{k}') \end{pmatrix}\end{aligned}$$

the last **polar axis basis matrix component**, zz' , being the same as the rectangular k -hat basis component, kk' . These being the same is an indicator of the equivalence between z -hat exponential (**expz**) and k -hat exponential (**expk**), since z -hat and k -hat are equivalent basis components and consequently $z\times$ and $k\times$ are equivalent complex numbers (and hence as are their basis exponentials **expz** and **expk**).

The component directions for the polar cylindrical basis-exponentials are written out and then diadically expanded, in the polar cylindrical basis, into the rotational matrix form for the polar cylindrical matrix basis:

For the **r -hat exponential**

$$\begin{aligned}r\text{-hat direction} & \exp(t\hat{r}\times(\cdot))\hat{r} = e^{t\hat{r}\times}\hat{r} = (\cos t)\hat{r} + (\sin t)\hat{r}\times\hat{r} = \hat{r} \\ \theta\text{-hat direction} & \exp(t\hat{r}\times(\cdot))\hat{\theta} = e^{t\hat{r}\times}\hat{\theta} = (\cos t)\hat{\theta} + (\sin t)\hat{r}\times\hat{\theta} = \hat{\theta}\cos t + \hat{z}\sin t \\ z\text{-hat direction} & \exp(t\hat{r}\times(\cdot))\hat{z} = e^{t\hat{r}\times}\hat{z} = (\cos t)\hat{z} + (\sin t)\hat{r}\times\hat{z} = \hat{z}\cos t - \hat{\theta}\sin t\end{aligned}$$

For the **θ -hat exponential**

$$\begin{aligned}r\text{-hat direction} & \exp(t\hat{\theta}\times(\cdot))\hat{r} = e^{t\hat{\theta}\times}\hat{r} = (\cos\theta t)\hat{r} + (\sin\theta t)\hat{\theta}\times\hat{r} = \hat{r}\cos t - \hat{z}\sin t \\ \theta\text{-hat direction} & \exp(t\hat{\theta}\times(\cdot))\hat{\theta} = e^{t\hat{\theta}\times}\hat{\theta} = (\cos\theta t)\hat{\theta} + (\sin\theta t)\hat{\theta}\times\hat{\theta} = \hat{\theta} \\ z\text{-hat direction} & \exp(t\hat{\theta}\times(\cdot))\hat{z} = e^{t\hat{\theta}\times}\hat{z} = (\cos\theta t)\hat{z} + (\sin\theta t)\hat{\theta}\times\hat{z} = \hat{z}\cos t + \hat{r}\sin t\end{aligned}$$

For the **z -hat exponential**

$$\begin{aligned}r\text{-hat direction} & \exp(t\hat{z}\times(\cdot))\hat{r} = e^{t\hat{z}\times}\hat{r} = (\cos z t)\hat{r} + (\sin z t)\hat{z}\times\hat{r} = \hat{r}\cos t + \hat{\theta}\sin t \\ \theta\text{-hat direction} & \exp(t\hat{z}\times(\cdot))\hat{\theta} = e^{t\hat{z}\times}\hat{\theta} = (\cos z t)\hat{\theta} + (\sin z t)\hat{z}\times\hat{\theta} = \hat{\theta}\cos t - \hat{r}\sin t \\ z\text{-hat direction} & \exp(t\hat{z}\times(\cdot))\hat{z} = e^{t\hat{z}\times}\hat{z} = (\cos z t)\hat{z} + (\sin z t)\hat{z}\times\hat{z} = \hat{z}\end{aligned}$$

The polar diadic expansions utilize $\hat{r}\hat{r}'=1$, $\hat{\theta}\hat{\theta}'=1$ and $\hat{z}\hat{z}'=1$ similarly as the principal rectangular diadic expansions utilizing $\hat{i}\hat{i}'=1$, $\hat{j}\hat{j}'=1$, and $\hat{k}\hat{k}'=1$ to do the expansion to the matrix form of the basis function [as an operator that operates on other basis function operators, and position and/or colour vectors].

Similarly as the rectangular system, the component directions diadically expand into the columns, in this instance, of the polar cylindrical matrix basis. For the r -hat exponential, the component directions are $e^{t\hat{r}\times}\hat{r} = \hat{r}$, $e^{t\hat{r}\times}\hat{\theta} = \hat{\theta}\cos t + \hat{z}\sin t$, and $e^{t\hat{r}\times}\hat{z} = \hat{z}\cos t - \hat{\theta}\sin t$ and hence

$$e^{t\hat{r}\times} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \Bigg|_{\text{polar basis}} = \begin{pmatrix} 1(\hat{r}\hat{r}') & 0(\hat{r}\hat{\theta}') & 0(\hat{r}\hat{z}') \\ 0(\hat{\theta}\hat{r}') & \cos t(\hat{\theta}\hat{\theta}') & -\sin t(\hat{\theta}\hat{z}') \\ 0(\hat{z}\hat{r}') & \sin t(\hat{z}\hat{\theta}') & \cos t(\hat{z}\hat{z}') \end{pmatrix}$$

Similarly, for the θ -hat exponential, the component directions are $e^{t\hat{\theta}\times}\hat{r} = \hat{r}\cos t - \hat{z}\sin t$, $e^{t\hat{\theta}\times}\hat{\theta} = \hat{\theta}$, and $e^{t\hat{\theta}\times}\hat{z} = \hat{z}\cos t + \hat{r}\sin t$, hence

$$e^{t\hat{\theta}\times} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix} \Bigg|_{\text{polar basis}} = \begin{pmatrix} \cos t(\hat{r}\hat{r}') & 0(\hat{r}\hat{\theta}') & \sin t(\hat{r}\hat{z}') \\ 0(\hat{\theta}\hat{r}') & 1(\hat{\theta}\hat{\theta}') & 0(\hat{\theta}\hat{z}') \\ -\sin t(\hat{z}\hat{r}') & 0(\hat{z}\hat{\theta}') & \cos t(\hat{z}\hat{z}') \end{pmatrix}$$

For the z -hat exponential, the component directions are $e^{t\hat{z}\times}\hat{r} = \hat{r}\cos t + \hat{\theta}\sin t$, $e^{t\hat{z}\times}\hat{\theta} = \hat{\theta}\cos t - \hat{r}\sin t$, and $e^{t\hat{z}\times}\hat{z} = \hat{z}$, hence

$$e^{t\hat{r}\times} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos t(\hat{r}\hat{r}') & -\sin t(\hat{r}\hat{\theta}') & 0(\hat{r}\hat{z}') \\ \sin t(\hat{\theta}\hat{r}') & \cos t(\hat{\theta}\hat{\theta}') & 0(\hat{\theta}\hat{z}') \\ 0(\hat{z}\hat{r}') & 0(\hat{z}\hat{\theta}') & 1(\hat{z}\hat{z}') \end{pmatrix} \text{polar basis}$$

These are the same as the rectangular basis-exponentials but in the polar matrix basis which is a good thing to know. This will be similarly for any generalized coordinate system and resulting basis and cross-product triangle, that is, each generalized coordinate system will have 3 rotation matrices (the basis exponentials) and these will have exactly the same form as the standard rotation matrices (**expk**, **expj**, and **expi**) when expressed in that basis, the basis defined by that coordinates system's **cross product triangle**.

Because, for cylindrical polars, the cylindrical axis is equivalent to one of the rectangular components [here the k -hat axis], the basis-exponentials for this axis are equivalent, here **expk** and **expz**. Therefore expanding out the the z -hat exponential **expz** in the polar matrix basis in terms of the standard rectangular basis components returns the k -hat exponential **expk**. This demonstrates that these are correct expressions obtained by stating the polar complex numbers after **cisk**-scaling rectangular complex numbers and then diadically expanding the basis exponentials. This is what I do here now, demonstrate the equivalence between **expk** and **expz** by expanding out the polar basis in terms of rectangulars. The z -hat polar vector basis is

$$\hat{r} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} = (\cos \theta \quad \sin \theta \quad 0)', \quad \hat{\theta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} = (-\sin \theta \quad \cos \theta \quad 0)', \text{ and } \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0 \quad 0 \quad 1)'$$

I think that similarly as colour vectors, the basis-hat exponentials applied to the colour vectors will be the bit-tracking equations for circularly cut Rubiks derivatives. Still to work out the details though. This also [possibly] means that there will be all of the equations of light (electromagnetic radiation) that can then be expressed in these different coordinate bases.

In the polar matrix basis, a matrix can be expressed in the expanded basis form, which is the same as in the matrix form with the basis components shown explicitly but in the form of an additive equation, the basis components essentially being place holders. This is similarly for the standard rectangular matrix basis. In the polar basis

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{polar basis} = \begin{pmatrix} \alpha_{11}\hat{r}\hat{r}' & \alpha_{12}\hat{r}\hat{\theta}' & \alpha_{13}\hat{r}\hat{z}' \\ \alpha_{21}\hat{\theta}\hat{r}' & \alpha_{22}\hat{\theta}\hat{\theta}' & \alpha_{23}\hat{\theta}\hat{z}' \\ \alpha_{31}\hat{z}\hat{r}' & \alpha_{32}\hat{z}\hat{\theta}' & \alpha_{33}\hat{z}\hat{z}' \end{pmatrix} = \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}'$$

and so can be expanded further and expressed in terms of the rectangular basis that has been scaled by the polars (the **cisk** scaling)

$$A = \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}'$$

$$A = \alpha_{11} \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_{12} \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_{13} \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} + \alpha_{21} \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_{22} \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ + \alpha_{23} \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} + \alpha_{31} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} + \alpha_{32} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} + \alpha_{33} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the matrix basis is rectangular and unshown at this stage because of the clutter. These are now in a form that can be expanded and terms collected. I will do this firstly for the z -hat exponential, to show equivalence to the k -hat exponential, as I have been saying I'll be doing [the above is the necessary algebra of the expansion], and then I'll do the same for the r -hat exponential and θ -hat exponential showing the full particulars of the circular scale factors when expressed in standard rectangulars. The polar radius is a rotated x or y axis rotated on the x - y plane orthogonal to z and then the r -hat exponential (**expr**, r -cross with its rotational eigenvector) is then a further orthogonal rotation around the radius vector r -hat which then meets the definition of a spherical.

$$\text{For } \mathbf{expz}, \text{ the } z\text{-hat basis exponential is } e^{t\hat{r}\times} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{polar basis} = \begin{pmatrix} \cos t(\hat{r}\hat{r}') & -\sin t(\hat{r}\hat{\theta}') & 0(\hat{r}\hat{z}') \\ \sin t(\hat{\theta}\hat{r}') & \cos t(\hat{\theta}\hat{\theta}') & 0(\hat{\theta}\hat{z}') \\ 0(\hat{z}\hat{r}') & 0(\hat{z}\hat{\theta}') & 1(\hat{z}\hat{z}') \end{pmatrix}$$

Expanding the polar matrix basis in terms of rectangular matrix basis components with the formula

$$A = \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}'$$

applied to the z -hat exponential **expz**

$$\begin{aligned}
e^{t\hat{r}\times} &= \cos t \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin t \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos t \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + 0 \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \cos t \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin t \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos t \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \cos t \begin{pmatrix} c_\theta^2 + s_\theta^2 & c_\theta s_\theta - s_\theta c_\theta & 0 \\ s_\theta c_\theta - c_\theta s_\theta & s_\theta^2 + c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} c_\theta s_\theta - s_\theta c_\theta & -c_\theta^2 - s_\theta^2 & 0 \\ s_\theta^2 + c_\theta^2 & -s_\theta c_\theta + c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

that is,

$$e^{t\hat{r}\times} = \cos t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos t(\hat{i}\hat{i}) & -\sin t(\hat{i}\hat{j}) & 0(\hat{i}\hat{k}) \\ \sin t(\hat{j}\hat{i}) & \cos t(\hat{j}\hat{j}) & 0(\hat{j}\hat{k}) \\ 0(\hat{k}\hat{i}) & 0(\hat{k}\hat{j}) & 1(\hat{k}\hat{k}) \end{pmatrix}$$

which is the k -hat exponential **expk** in the rectangular matrix basis. Therefore, the **equivalence has been demonstrated**, that is, one-for-one equivalence between z -hat **expz** and k -hat exponential **expk** in the respective matrix bases. This equivalence (**expz** = **expk**) is summarized here in the polar basis and then in the rectangular basis showing equivalence

$$\mathbf{expz} = e^{t\hat{r}\times} = \begin{pmatrix} \cos t(\hat{r}\hat{r}) & -\sin t(\hat{r}\hat{\theta}) & 0(\hat{r}\hat{z}) \\ \sin t(\hat{\theta}\hat{r}) & \cos t(\hat{\theta}\hat{\theta}) & 0(\hat{\theta}\hat{z}) \\ 0(\hat{z}\hat{r}) & 0(\hat{z}\hat{\theta}) & 1(\hat{z}\hat{z}) \end{pmatrix} = \begin{pmatrix} \cos t(\hat{i}\hat{i}) & -\sin t(\hat{i}\hat{j}) & 0(\hat{i}\hat{k}) \\ \sin t(\hat{j}\hat{i}) & \cos t(\hat{j}\hat{j}) & 0(\hat{j}\hat{k}) \\ 0(\hat{k}\hat{i}) & 0(\hat{k}\hat{j}) & 1(\hat{k}\hat{k}) \end{pmatrix} = e^{t\hat{k}\times} = \mathbf{expk}$$

expz z -hat exponential = zz' rotation matrix in cylindrical basis = kk' rotation matrix rectangular basis = k -hat exponential **expk**

This, equivalence between **expk** and **expz**, demonstrates that the maths is legitimate and works as it should and so it is possible to do both inner products (tensor order contraction) and outer products (tensor order expansion) in polar vector bases, for diadic expansions for example, similarly as for rectangular vector bases. Quite pleased actually since I wasn't sure how the diadic expansions would work in polar bases, or if it would be possible at all. This demonstrates clearly that it is and it all works as I had hoped. Pretty hard going mathematically though, and one would hope that there is an easier way and it turns out that there is, change of base which I will discuss in the next and last topic of this document.

The axes of the orthogonal complement to the cylindrical rotation axis is the basis vectors r -hat and q -hat of the polar plane and these are rotated vectors of i -hat and j -hat so the basis exponentials of $r \times$ and $\theta \times$ (**expr** and **expq**) are distinct from those of $i \times$ and $j \times$ (**expi** and **expj**) when expanded out in terms of the standard matrix basis. These differ in terms of the polar scale factors, that is, the harmonics of the scaling **expk** = **cisk** basis exponential. These are expanded now, to show their polar forms in terms of the standard rectangular bases.

$$\text{For } \mathbf{expr}, \text{ the } r\text{-hat exponential, } e^{t\hat{r}\times} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \underset{\text{polar basis}}{=} \begin{pmatrix} 1(\hat{r}\hat{r}) & 0(\hat{r}\hat{\theta}) & 0(\hat{r}\hat{z}) \\ 0(\hat{\theta}\hat{r}) & \cos t(\hat{\theta}\hat{\theta}) & -\sin t(\hat{\theta}\hat{z}) \\ 0(\hat{z}\hat{r}) & \sin t(\hat{z}\hat{\theta}) & \cos t(\hat{z}\hat{z}) \end{pmatrix}$$

Similarly as for **expz**, expanding **expr** in the **polar matrix basis** in terms of **rectangular matrix basis components** with the formula

$$A = \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}'$$

The r -hat exponential **expr** is then

$$\begin{aligned}
e^{t\hat{r}\times} &= 1 \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos t \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad - \sin t \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= 1 \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos t \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos t \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 0 & s_\theta \\ 0 & 0 & -c_\theta \\ -s_\theta & c_\theta & 0 \end{pmatrix} \\
&= \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} s_\theta^2 \cos t & -s_\theta c_\theta \cos t & 0 \\ -s_\theta c_\theta \cos t & c_\theta^2 \cos t & 0 \\ 0 & 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_\theta \sin t \\ 0 & 0 & -c_\theta \sin t \\ -s_\theta \sin t & c_\theta \sin t & 0 \end{pmatrix} \\
&= \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} s_\theta^2 \cos t & -s_\theta c_\theta \cos t & 0 \\ -s_\theta c_\theta \cos t & c_\theta^2 \cos t & 0 \\ 0 & 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_\theta \sin t \\ 0 & 0 & -c_\theta \sin t \\ -s_\theta \sin t & c_\theta \sin t & 0 \end{pmatrix}
\end{aligned}$$

that is,

$$\begin{aligned}
e^{t\hat{r}\times} &= \begin{pmatrix} 1(\hat{r}\hat{r}') & 0(\hat{r}\hat{\theta}') & 0(\hat{r}\hat{z}') \\ 0(\hat{\theta}\hat{r}') & \cos t(\hat{\theta}\hat{\theta}') & -\sin t(\hat{\theta}\hat{z}') \\ 0(\hat{z}\hat{r}') & \sin t(\hat{z}\hat{\theta}') & \cos t(\hat{z}\hat{z}') \end{pmatrix} = \begin{pmatrix} (c_\theta^2 + s_\theta^2 \cos t)\hat{i}\hat{i}' & (c_\theta s_\theta - s_\theta c_\theta \cos t)\hat{i}\hat{j}' & (s_\theta \sin t)\hat{i}\hat{k}' \\ (s_\theta c_\theta - s_\theta c_\theta \cos t)\hat{j}\hat{i}' & (s_\theta^2 + c_\theta^2 \cos t)\hat{j}\hat{j}' & (-c_\theta \sin t)\hat{j}\hat{k}' \\ (-s_\theta \sin t)\hat{k}\hat{i}' & (c_\theta \sin t)\hat{k}\hat{j}' & (\cos t)\hat{k}\hat{k}' \end{pmatrix} \\
&= \begin{pmatrix} c_\theta^2 + s_\theta^2 \cos t & c_\theta s_\theta - s_\theta c_\theta \cos t & s_\theta \sin t \\ s_\theta c_\theta - s_\theta c_\theta \cos t & s_\theta^2 + c_\theta^2 \cos t & -c_\theta \sin t \\ -s_\theta \sin t & c_\theta \sin t & \cos t \end{pmatrix}
\end{aligned}$$

which is a complicated expression, as expressed in the standard rectangular basis for the polar-radius r -hat exponential **expr**, noting that whilst in the polar basis it has the exact same form as the i -hat exponential **exp*i***. On the one hand, it is a bit disappointing it is complicated, but on the other it is good since it means that there is a lot of additional dynamic over the standard basis-exponentials [in this case the i -hat basis exponential **exp*i***] for which will make this difficult mathematics simple by simply working with the r -hat exponential **expr** itself in the polar basis [with all this maths in this basis the same pretty much as though working with the i -hat exponential **exp*i*** in the rectangular basis and the first component ($r\times$ rather than $i\times$) in the respective cross product triangle]. Probably the most interesting application will be a circularly polarised vector of light along the radius vector. This would be the slinky model of a pulse of radar energy and its return echo along an arbitrary radius vector, the return echo being the double cross r -hat ($r \times r \times$) exponential **exprr**. What I show in a few pages is that this is a **change of principal axis of rotation (change of basis) matrix** for an arbitrary azimuthal angle. In the document that I called *change of principal axis of rotation* I changed the k -hat, j -hat, and/or i -hat cross products ($k\times$, $j\times$, $i\times$) of the basis sines, cosines, and exponentials (such as **sink**, **cosk**, and **expk**) to alternative principal bases (such as to **sinj**, **cosj**, and **expj**) such as **sink**→**sinj**, **cosk**→**cosj**, and **expk**→**expj** with the **exp*i*** $\pi/2 = \text{cisi } \pi/2$ (quarter-turn) exponential [the full rank complex number for i -hat x -rotation axis] as the change of base [principal rotation axis] matrix. I noted that it is possible to change the bases to non-principal axes for arbitrary rotations around one of the principal rectangular axes. That is what the above formula does via the transformation to cylindricals for an arbitrary cylindrical polar angle θ . **More complicated** (than for $\theta = \pi/2$ and π) but **more general** (for arbitrary θ). The reason for doing so is two-fold. One can arbitrarily rotate the system to a new orientation to work in (**derived schleronomic basis**) or can set the system spinning at a time dependent angle/frequency (**derived rheonomic basis**) and work in a system with a continually changing change of base, for example, on a spinning Earth. The former is a **schleronomic change of base** (hence a derivative base, derived from rectangulars, derived schleronomic) while the latter is a **rheonomic change of base (derived rheonomic)** so that one has to be mindful when doing any time calculus (time integrals or derivatives) and this mathematical nomenclature provides for making this distinction, that is, mathematical nomenclature [adapted from Lagrange theory and derivatives of finance and carboxylic acids] for whether the theta-argument of the **cisk**-scaling that transforms the system is constant or time dependent [and so more likely not to blunder and forget about the Coriolis force for example],

$$\theta = \text{const. [derived scleronomic]} \quad \text{or} \quad \theta = \omega t \text{ [derived rheonomic]} \quad \text{for} \quad \begin{array}{ccc} \hat{i} & & \hat{r} \\ \nearrow & \searrow & \nearrow \\ \hat{k} & \leftarrow \hat{j} & \hat{z} \end{array} \xrightarrow{\text{cisk}\theta} \begin{array}{ccc} \hat{r} & & \hat{\theta} \\ \nearrow & \searrow & \nearrow \\ \hat{z} & \leftarrow \hat{\theta} & \end{array}$$

The power of words. This mathematical nomenclature, **derived schleronomic** (single change of orientation) and **derived rheonomic** (spinning) is going to prove [possibly] to be very worthwhile: again, **derived schleronomic change of base** (**cisk** scaling with $\theta = \text{const.}$) and **derived rheonomic change of base** (**cisk** scaling with $\theta = \omega t$).

Since the maths for expanding out the polar basis exponentials in the polar matrix basis in terms of the rectangular basis is somewhat lengthy, with ample scope for muddling things up [especially when being worked out for the first time], if one is interested to do the maths for themselves (and avoid tearing their hair out in frustration) the long way [rather than a change of base on a TI that I discuss

shortly], I simply put the expansion of a matrix expanded in terms of the polar basis for arbitrary components and then substituted in the components along subsequent rows and made sure that they corresponded with the other polar exponentials [I wrote the basis out on paper and then typed it up and substituted]. I do this again in the next bit of working for the θ -hat exponential **expq** similarly as for the z -hat and r -hat exponentials, **expz** and **expr**. One could probably appreciate why I found it good to write with the further abbreviated forms of sine and cosine [to s and c from $\sin$ and $\cos$]. I might make this more standard in my work, especially when scaling vector and matrix bases [used it quite a few times in exams and assignments as of late (last time in linear algebra... I have sat aaaaaa lot of linear algebra... don't actually mind because I love the subject)]. Expanding out the remaining polar basis exponential in terms of the standard rectangular basis:

$$\text{For } \mathbf{expq}, \text{ the } \theta\text{-hat exponential, } e^{t\hat{\theta}^\times} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}^{\text{polar basis}} = \begin{pmatrix} \cos t(\hat{r}\hat{r}') & 0(\hat{r}\hat{\theta}') & \sin t(\hat{r}\hat{z}') \\ 0(\hat{\theta}\hat{r}') & 1(\hat{\theta}\hat{\theta}') & 0(\hat{\theta}\hat{z}') \\ -\sin t(\hat{z}\hat{r}') & 0(\hat{z}\hat{\theta}') & \cos t(\hat{z}\hat{z}') \end{pmatrix}$$

Similarly as for **expz** and **expr**, expanding the **polar matrix basis** in terms of **rectangular matrix basis components** with the formula

$$A = \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}'$$

The θ -hat exponential **expq** is then

$$\begin{aligned} e^{t\hat{\theta}^\times} &= \cos t \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} -c_\theta s_\theta & c_\theta^2 & 0 \\ -s_\theta^2 & s_\theta c_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} -s_\theta c_\theta & -s_\theta^2 & 0 \\ c_\theta^2 & c_\theta s_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + 0 \begin{pmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_\theta & c_\theta & 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \cos t \begin{pmatrix} c_\theta^2 & c_\theta s_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 0 & c_\theta \\ 0 & 0 & s_\theta \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_\theta & s_\theta & 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\theta^2 \cos t & c_\theta s_\theta \cos t & 0 \\ s_\theta c_\theta \cos t & s_\theta^2 \cos t & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_\theta \sin t \\ 0 & 0 & s_\theta \sin t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c_\theta \sin t & -s_\theta \sin t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos t \end{pmatrix} \\ &= \begin{pmatrix} c_\theta^2 \cos t & c_\theta s_\theta \cos t & 0 \\ s_\theta c_\theta \cos t & s_\theta^2 \cos t & 0 \\ 0 & 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_\theta \sin t \\ 0 & 0 & s_\theta \sin t \\ -c_\theta \sin t & -s_\theta \sin t & 0 \end{pmatrix} + \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta & 0 \\ -s_\theta c_\theta & c_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

that is,

$$e^{t\hat{\theta}^\times} = \begin{pmatrix} c_\theta^2 \cos t + s_\theta^2 & c_\theta s_\theta \cos t - s_\theta c_\theta & c_\theta \sin t \\ s_\theta c_\theta \cos t - s_\theta c_\theta & s_\theta^2 \cos t + c_\theta^2 & s_\theta \sin t \\ -c_\theta \sin t & -s_\theta \sin t & \cos t \end{pmatrix} = \begin{pmatrix} (c_\theta^2 \cos t + s_\theta^2)\hat{i}\hat{i}' & (c_\theta s_\theta \cos t - s_\theta c_\theta)\hat{i}\hat{j}' & (c_\theta \sin t)\hat{i}\hat{k}' \\ (s_\theta c_\theta \cos t - s_\theta c_\theta)\hat{j}\hat{i}' & (s_\theta^2 \cos t + c_\theta^2)\hat{j}\hat{j}' & (s_\theta \sin t)\hat{j}\hat{k}' \\ (-c_\theta \sin t)\hat{k}\hat{i}' & (-s_\theta \sin t)\hat{k}\hat{j}' & (\cos t)\hat{k}\hat{k}' \end{pmatrix}$$

Seemed like very hard work (extensive algebra) and it is a complicated expression expressed in rectangulars. Is there an easier way? There is not a pressing need to express the polar basis exponentials in terms of rectangulars once their form is known, but yeah, there is a better way when it is necessary to do so. This is a change of basis from a principal rotation axis ($k\times, j\times, i\times$) to arbitrary rotation axis (from **cisk**, **ciskj**, or **cisi** scaling the rectangular system with one of the rectangular basis exponentials). The change of basis from standard rectangulars to cylindrical polars is up for discussion next.

In the discussion of a change of basis that converts a basis-sine from one principal axis to another, say **sink** to **sinj**, by means of a rotation about the i -hat axis by application of **cisi** on $\pi/2$ is what the above formula does, but for an arbitrary rotational angle θ . For that document, the base change was between principal axis for when $\theta = \pi/2$. [a **derived schleronomic change of base**] to change between two principal axes but here the rotation angle θ of the **cisk**-scaling that changes the base is arbitrary, and hence the complicated forms of the formulae. One can see that they simplify by putting $\theta = \pi/2$ into the scale factors s_θ and c_θ , the latter being 0 and the former 1 so that the above θ -hat basis exponential **expq** converts to the principal rotation axis of i -hat-conjugate and so that the expression simplifies to that of **expi'**, i.e., the change of base of principal rotation axes **expj** $\rightarrow$ **expi'**.

This is similarly for the basis-cosine, basis-exponential, and the complex number for that principal axis, that is, converts $\hat{k} \times$ to $\hat{j} \times$ or $\hat{i} \times$ by change of principal rotation axis when θ is a multiple of $\pi/2$ [which means that the change of base is orthogonal and hence between principal rotation axes].

The change of principal axis discussion whereby the basis cross ($k \times$), exponential (**expk**), and harmonics (**cosk**, **sink**, and **tank**) for a k -hat principal axis have the principal axis changed from this axis to one of the other orthogonal rectangular axes, were change of basis for theta rotations at multiple of $\pi/2$, that is, for orthogonal change of basis to one of the other principal axes. Naturally, I had also considered what would the formula be if the change of principal axis had been for some arbitrary rotation, that is, for an arbitrary theta rotation rather than some multiple of $\pi/2$. The reason that I didn't discuss this at the time was that the resulting equations, that is, the resulting rotation matrix for the change of base (**expi** as the change of basis matrix), resulting in expressions that were too complicated (for arbitrary θ). Keeping the formulae on the principal axes made for a nice discussion that could be understood intuitively, that is, changing the base of the complex number $k \times$ to $j \times$ and all of the basis functions generated by $k \times$, which all changed over to that of $j \times$ similarly. So naturally the question then was *so what were the complicated formulae for change of basis to non-principal axes?* As it turns out, not so surprisingly, that these formulae are the ones just obtained for the polar basis functions generated by the polar complex numbers $r \times$, $q \times$, and $z \times$, including the complicated expressions of the r -hat and θ -hat exponentials, **expr** and **expq**, when they are expressed, here [in the discussion of polar bases], in the underlying standard rectangular coordinates after expanding out the polar matrix basis. Hence, these can be explained intuitively here and the underlying complexity [in terms of analytic difficulty] of arbitrary change of basis in rectangulars, for example, when not between principal axes, can be overcome by simply working in the different matrix basis with the associated polar complex numbers (r -cross [$r \times$], θ -cross [$\theta \times$], and z -cross [$z \times$]), that is, in the different coordinate system. This seems obvious now but in the past has been easier said than done without all of the formulae having been worked out here [and demonstrating that it all works]. The polar complex numbers (r -cross [$r \times$], θ -cross [$\theta \times$], and z -cross [$z \times$]), and their basis-exponentials (r -hat exponential [**expr**], θ -hat exponential [**expq**], and z -hat exponential [**expz**]) provides a means to do this as easily as working in standard rectangulars. To do this, then, one scales the rectangular system with the relevant cis-scaling [**expk=cisk**, **expj=cisj**, or **expi=cisi**] and then works in the scaled basis in the exact same way as working in the rectangular system. And when one wants to go back and forth between the two systems, for any particular reason, one can use the change of base formulae which I will be now discussing. This is also similarly for a polar spherical system that is scaled with two successive cis-scalings, **cisk** and then **cisj** for example. So far, it has all worked out nicely and the different parts of what I have been working out in this latest addition to the theory of $k \times$ [k -cross] is all working out to be consistent when extending to the polar complex number $r \times$ [r -cross] and quaternion set ($r \times$, $\theta \times$, and $z \times$), and all of the associated basis functions.

I think what I have managed to do here is make a full union between complex analysis and linear algebra (now in generalized coordinate systems). What I am really hoping for now is that I will be able to sort it all when doing calculus with it, especially for the derived rheonomic systems that are spinning. This will be the full union with vector calculus that I am seeking [which I then hope to turn into time-series and transfer functions in a Gauss basis].

SECTION 4 – Change of base

The change of base from rectangular to cylindrical scales the rectangular system by the basis exponential that corresponds to the cylindrical axis of the cylindrical system, that is, the pole of the polar system. If this is the z-axis, which is generated by the basis vector $\hat{k}$, then the rectangular system is scaled by the $\hat{k}$ -basis exponential $\mathbf{expk} = \mathbf{cisk}$. Since $\mathbf{expk}$ is the $\hat{k}$ -hat rotation matrix, the change of base scaling rotates the rectangular system into the new orientation, the scaling rotation being brought about by the basis harmonics of $\mathbf{cosk}$ and $\mathbf{sink}$ contained in $\mathbf{cisk}$. The system is still rectangular inasmuch as the axes are still orthogonal, the difference being, in addition to converting the initial position vector to magnitude and phase, that the x and y axes point in different directions according to the cylindrical scaling/rotating angle θ of the change of base basis exponential. The change-of-base exponential, as a diadically expanded cross-product (complex number) operator [not a number or scalar], is the change of basis matrix for the change-of-base transformation to the new coordinates. This essentially is the utilization of the complex exponential to change base. The change-of-base matrix P is that defined for the similarity relation, most frequently for the diagonalization process, $A \sim B$ if

$A = PBP^{-1}$, here, $P = e^{\theta\hat{k}\times}$. This demonstrates that the diagonalization procedure *in this context* is a change of base from the principal rectangular axes to those defined by the polar system, cylindrical or spherical. This makes the procedure intuitive by the use of $\mathbf{expk}$, and means that the $\hat{k}$ -basis exponential (complex exponential) could equivalently be referred to as a change of base exponential. The change of base with $\mathbf{expk}$ can be further rotated for spherical systems and arbitrary Euler angles and direction cosines, with $\mathbf{expi}$ and/or $\mathbf{expj}$, for example $P = e^{\theta\hat{k}\times}e^{\phi\hat{j}\times}$ and $P^{-1} = e^{-\phi\hat{j}\times}e^{-\theta\hat{k}\times}$ for the linear algebra change of base formula $A = PBP^{-1}$ to sphericals. Listing of all the possible polar axes changes of base to cylindricals

$P = e^{\theta\hat{k}\times}$	with $\mathbf{expk}$ changes base to ccw z-axis polar cylindricals
$P = (e^{\theta\hat{k}\times})' = e^{-\theta\hat{k}\times}$	with $\mathbf{expk}'$ changes base to cw z-axis polar cylindricals
$P = e^{\theta\hat{j}\times}$	with $\mathbf{expj}$ changes base to ccw y-axis polar cylindricals
$P = (e^{\theta\hat{j}\times})' = e^{-\theta\hat{j}\times}$	with $\mathbf{expj}'$ changes base to cw y-axis polar cylindricals
$P = e^{\theta\hat{i}\times}$	with $\mathbf{expi}$ changes base to ccw x-axis polar cylindricals
$P = (e^{\theta\hat{i}\times})' = e^{-\theta\hat{i}\times}$	with $\mathbf{expi}'$ changes base to cw x-axis polar cylindricals

Change of base with $\mathbf{expk}$ to polar cylindricals

Note at the outset here that what you get with the change of base formula $e^{\theta\hat{k}\times}\mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times)e^{-\theta\hat{k}\times} = \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times)$ [this formula is the purpose of this section] is the polar basis functions expressed in terms of polar bases expanded out in the standard rectangular basis, as per the hard going mathematical workings of the previous section. If these are not expanded out in terms of the standard basis, then the basis functions are exact analogues of those in the standard basis but expressed in terms of $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat and so only work on, that is operate on, vectors of $\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat. Important to bear this in mind [this is probably the most confusing item in second year university linear algebra as well... that is, keeping track of which vector and matrix base you are working in when there are changes of base].

Each component in a rectangular matrix basis can be individually transformed to the polar cylindrical matrix [or polar spherical matrix basis]. Considering each component like this shows how the change of base works. Considering the ii' component, the first diagonal component in the standard rectangular matrix basis. Applying the $\hat{k}$ -basis exponential $\mathbf{expk}$ as the change of basis rotation matrix

$$e^{\theta\hat{k}\times}\hat{i}\hat{i}'e^{-\theta\hat{k}\times} = \hat{r}\hat{r}'$$

which then results in the first diagonal element in the **ccw** polar cylindrical matrix basis and then expresses this component of the polar basis in terms of the rectangular matrix basis and the scale factors from the $\mathbf{expk}$ rotating the system. Makes short work of the overly complicated maths. Demonstrating that it works:

$$e^{\theta\hat{k}\times}\hat{i}\hat{i}'e^{-\theta\hat{k}\times} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{r}\hat{r}'$$

noting that since $\hat{r} = e^{\theta\hat{k}\times}\hat{i} = \hat{i}\cos\theta + \hat{j}\sin\theta$ is the $\hat{i}$ -hat component of $\mathbf{expk}$ and hence equivalent to $\hat{r}$ -hat as **cisk**-scaled $\hat{i}$ -hat

$$\hat{r}\hat{r}' = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The inverse of the change of basis also applies so that

$$\hat{i}\hat{i}' = e^{-\theta\hat{k} \times} \hat{r}\hat{r}' e^{\theta\hat{k} \times}$$

With these matrices, if it works one way it works the other. Demonstrating anyway that the inverse change of base works:

$$e^{-\theta\hat{k} \times} \hat{r}\hat{r}' e^{\theta\hat{k} \times} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{i}\hat{i}'$$

These formulae are in the format that defines similarity. Similarity is defined for purposes of raising to powers and hence matrix exponentials. Matrix exponentials are basis exponentials and hence also complex exponentials. Both forward and backward changes of base can be proved also with the Euler **cisk** form of **expk**. Noting that writing the conjugate basis exponential **expk'** with a negative exponent is a mathematical convenience. What I am of the opinion of at the moment is that it is not best described as negative but

rather as a reversed cross product triangle so that $(e^{\theta\hat{k} \times})' = e^{\times\hat{k}\theta}$. For the change of base to polars, proving by **expk** = **cisk**,

$$\begin{aligned} e^{\theta\hat{k} \times} \hat{i}\hat{i}' e^{\times\hat{k}\theta} &= (\mathbf{cisk} \theta + \mathbf{sink} \theta \hat{k} \times) \hat{i}\hat{i}' (\mathbf{cisk}' \theta + \times \hat{k}' \mathbf{sink}' \theta) \\ &= (\mathbf{cisk} \theta \hat{i} + \mathbf{sink} \theta \hat{k} \times \hat{i}) (\hat{i}' \mathbf{cisk}' \theta + \hat{i}' \times \hat{k}' \mathbf{sink}' \theta) \\ &= (\mathbf{cisk} \theta \hat{i} + \mathbf{sink} \theta \hat{j}) (\hat{i}' \mathbf{cisk}' \theta - \hat{j}' \mathbf{sink}' \theta) \\ &= (\hat{i} \cos \theta + \hat{j} \sin \theta) (\cos \theta \hat{i}' - \sin \theta \hat{j}') \\ &= \hat{i} \cos \theta (\cos \theta \hat{i}' - \sin \theta \hat{j}') + \hat{j} \sin \theta (\cos \theta \hat{i}' - \sin \theta \hat{j}') \\ &= \hat{i}\hat{i}' \cos^2 \theta - \hat{i}\hat{j}' \cos \theta \sin \theta + \hat{j}\hat{i}' \sin \theta \cos \theta - \hat{j}\hat{j}' \sin^2 \theta \\ &= \hat{r}\hat{r}' \end{aligned}$$

and similarly for the inverse change of base since the basis exponential is full rank.

There are two ways to consider the change of base $e^{\theta\hat{k} \times} \hat{i}\hat{i}' e^{\times\hat{k}\theta}$ intuitively [same maths, slightly different intuition] and are essentially the qualitative reasoning behind the above proofs:

(1) The first is to consider the first column basis vector being selected from the left exponential **expk** by the first standard rectangular component i in ii' (giving $e^{\theta\hat{k} \times} \hat{i} = \hat{r}$) and the first row vector from the right basis exponential **expk'** being selected by the primed rectangular component (giving $\hat{i}' e^{-\theta\hat{k} \times} = \hat{r}'$) and the two then expanded into the polar matrix basis component by outer product. $\hat{i}$ is the standard rectangular basis component and $\hat{i}\hat{i}'$ is the standard rectangular matrix basis component, $e^{\theta\hat{k} \times} \hat{i}$ is the cylindrical column basis vector $\hat{r}$, when primed becomes $(e^{\theta\hat{k} \times} \hat{i})' = \hat{i}' (e^{\theta\hat{k} \times})' = \hat{i}' e^{-\theta\hat{k} \times}$ which is the cylindrical row vector $\hat{r}'$ which then on outer product return the basis component in the new scaled/rotated coordinates.

$$\begin{array}{ccc} \begin{array}{c} \hat{i} \\ \nearrow \quad \searrow \\ \hat{k} \quad \leftarrow \quad \hat{j} \end{array} & \xrightarrow{\mathbf{cisk} \theta} & \begin{array}{c} \hat{r} \\ \nearrow \quad \searrow \\ \hat{z} \quad \leftarrow \quad \hat{\theta} \end{array} \end{array} \quad \text{and priming} \quad \begin{array}{ccc} \begin{array}{c} \hat{i}' \\ \nwarrow \quad \swarrow \\ \hat{k}' \rightarrow \hat{j}' \end{array} & \xrightarrow{\mathbf{cisk}' \theta} & \begin{array}{c} \hat{r}' \\ \nwarrow \quad \swarrow \\ \hat{z}' \rightarrow \hat{\theta}' \end{array} \end{array}$$

and so

$$\begin{array}{ccc} \begin{array}{c} e^{\theta\hat{k} \times} \hat{i} \\ \nearrow \quad \searrow \\ e^{\theta\hat{k} \times} \hat{k} \quad \leftarrow \quad e^{\theta\hat{k} \times} \hat{j} \end{array} & \times & \begin{array}{c} \hat{i}' e^{-\theta\hat{k} \times} \\ \nwarrow \quad \swarrow \\ \hat{k}' e^{-\theta\hat{k} \times} \quad \leftarrow \quad \hat{j}' e^{-\theta\hat{k} \times} \end{array} \\ & = & \begin{array}{c} \hat{r} \\ \nearrow \quad \searrow \\ \hat{z} \quad \leftarrow \quad \hat{\theta} \end{array} \times \begin{array}{c} \hat{r}' \\ \nwarrow \quad \swarrow \\ \hat{z}' \rightarrow \hat{\theta}' \end{array} \end{array}$$

(2) The second is to take the rectangular matrix basis component, the column component i in ii' , and rotate it into the column component of the new orientation (with the left-multiplier rotation basis exponential **expk** giving $e^{\theta\hat{k} \times} \hat{i} = \hat{r}$) and take the second matrix basis component, the row component i' in ii' , and similarly rotate it into the row component of the new orientation (with the right-multiplier conjugate basis exponential **expk'** giving $\hat{i}' e^{-\theta\hat{k} \times} = \hat{r}'$) which then returns the polar matrix basis component upon performing the outer product.

This was an attempt at least to provide intuition behind the formula $e^{\theta\hat{k} \times} \mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) e^{-\theta\hat{k} \times} = \mathbf{p}(\hat{r} \times, \hat{\theta} \times, \hat{z} \times)$, the **expk** change of base formula for which is the purpose of this section.

The diagonal elements (rr' , $\theta\theta'$, and zz'), those that are utilized to diadically expand, are idempotent and can be raised to [null] powers and hence fit the definition of similarity. This is interesting because these matrices, expressed in rectangulars [which retain idempotence], are more examples of similar matrices that are not diagonal or in Jordan form [when expanded into the rectangular basis], which is not stated arrogantly, but rather as a statement that this theory is going to provide us with more mathematical resolving power. It is obvious that ii' is idempotent because it is an identity diagonal element and because the transpose is used to diadically expand with $i'i = 1$, so $(ii')(ii') = i(i'i)i' = i(1)i' = ii'$ so that $(ii')^n = ii'$ and similarly for rr' since it is similar $ii' \sim rr'$ so $(rr')(rr') = r(r'r)r' = r(1)r' = rr'$ so that $(rr')^n = rr'$. In matrix form,

$$\begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

showing idempotence of a rather complicated matrix [compared to the elements of the identity in the standard matrix basis and especially complicated in the context of all 9 polar matrix basis components taken together]. The basis components of the two band diagonals are nilpotent of order 1 and the remaining two in the non-diagonal corners are nilpotent of order 0, for both rectangulars and for the derived system because the system remains orthogonal. With the other basis components, the full set of polar matrix basis components obtained by change of base with **expk** are

$$\begin{aligned} e^{\theta\hat{k}\times} \hat{i}\hat{i}' e^{-\theta\hat{k}\times} &= \hat{r}\hat{r}' & e^{\theta\hat{k}\times} \hat{i}\hat{j}' e^{-\theta\hat{k}\times} &= \hat{r}\hat{\theta}' & e^{\theta\hat{k}\times} \hat{i}\hat{k}' e^{-\theta\hat{k}\times} &= \hat{r}\hat{z}' \\ e^{\theta\hat{k}\times} \hat{j}\hat{i}' e^{-\theta\hat{k}\times} &= \hat{\theta}\hat{r}' & e^{\theta\hat{k}\times} \hat{j}\hat{j}' e^{-\theta\hat{k}\times} &= \hat{\theta}\hat{\theta}' & e^{\theta\hat{k}\times} \hat{j}\hat{k}' e^{-\theta\hat{k}\times} &= \hat{\theta}\hat{z}' \\ e^{\theta\hat{k}\times} \hat{k}\hat{i}' e^{-\theta\hat{k}\times} &= \hat{z}\hat{r}' & e^{\theta\hat{k}\times} \hat{k}\hat{j}' e^{-\theta\hat{k}\times} &= \hat{z}\hat{\theta}' & e^{\theta\hat{k}\times} \hat{k}\hat{k}' e^{-\theta\hat{k}\times} &= \hat{z}\hat{z}' \end{aligned}$$

which can be similarly demonstrated that they work with their matrix form, or alternatively by complex number theory of power series expansions of k -cross, $k\times$, or with the Euler formula **expk** = **cisk** for the basis exponentials [and the same intuition]. This shows again that matrices are essentially systems of complex numbers, the additional theory required to see this being that priming differentiates a new set of three complex numbers [with a reverse cross-product triangle according to my current opinion, the negatives are for convenience and result simply from the reversing of direction of the cross-product triangle] and this forms the set for the matrix basis as the outer product. Differentiating between rows and columns is very important and is definitely not trivial, including when appended with $\times$. In the rectangular matrix basis

$$\mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \alpha_{11}\hat{i}\hat{i}' + \alpha_{12}\hat{i}\hat{j}' + \alpha_{13}\hat{i}\hat{k}' + \alpha_{21}\hat{j}\hat{i}' + \alpha_{22}\hat{j}\hat{j}' + \alpha_{23}\hat{j}\hat{k}' + \alpha_{31}\hat{k}\hat{i}' + \alpha_{32}\hat{k}\hat{j}' + \alpha_{33}\hat{k}\hat{k}'$$

Hence, the change of basis for the $3 \times 3 = 9$ component matrix basis **p** is $e^{\theta\hat{k}\times} \mathbf{p} e^{-\theta\hat{k}\times}$

$$\begin{aligned} e^{\theta\hat{k}\times} \mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) e^{-\theta\hat{k}\times} &= e^{\theta\hat{k}\times} \left(\alpha_{11}\hat{i}\hat{i}' + \alpha_{12}\hat{i}\hat{j}' + \alpha_{13}\hat{i}\hat{k}' + \alpha_{21}\hat{j}\hat{i}' + \alpha_{22}\hat{j}\hat{j}' + \alpha_{23}\hat{j}\hat{k}' + \alpha_{31}\hat{k}\hat{i}' + \alpha_{32}\hat{k}\hat{j}' + \alpha_{33}\hat{k}\hat{k}' \right) e^{-\theta\hat{k}\times} \\ &= \alpha_{11}e^{\theta\hat{k}\times} \hat{i}\hat{i}' e^{-\theta\hat{k}\times} + \alpha_{12}e^{\theta\hat{k}\times} \hat{i}\hat{j}' e^{-\theta\hat{k}\times} + \alpha_{13}e^{\theta\hat{k}\times} \hat{i}\hat{k}' e^{-\theta\hat{k}\times} \\ &\quad + \alpha_{21}e^{\theta\hat{k}\times} \hat{j}\hat{i}' e^{-\theta\hat{k}\times} + \alpha_{22}e^{\theta\hat{k}\times} \hat{j}\hat{j}' e^{-\theta\hat{k}\times} + \alpha_{23}e^{\theta\hat{k}\times} \hat{j}\hat{k}' e^{-\theta\hat{k}\times} \\ &\quad + \alpha_{31}e^{\theta\hat{k}\times} \hat{k}\hat{i}' e^{-\theta\hat{k}\times} + \alpha_{32}e^{\theta\hat{k}\times} \hat{k}\hat{j}' e^{-\theta\hat{k}\times} + \alpha_{33}e^{\theta\hat{k}\times} \hat{k}\hat{k}' e^{-\theta\hat{k}\times} \\ &= \alpha_{11}\hat{r}\hat{r}' + \alpha_{12}\hat{r}\hat{\theta}' + \alpha_{13}\hat{r}\hat{z}' + \alpha_{21}\hat{\theta}\hat{r}' + \alpha_{22}\hat{\theta}\hat{\theta}' + \alpha_{23}\hat{\theta}\hat{z}' + \alpha_{31}\hat{z}\hat{r}' + \alpha_{32}\hat{z}\hat{\theta}' + \alpha_{33}\hat{z}\hat{z}' \\ &= \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times) \end{aligned}$$

that is,

$$e^{\theta\hat{k}\times} \mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) e^{-\theta\hat{k}\times} = e^{\theta\hat{k}\times} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} e^{-\theta\hat{k}\times} = e^{\theta\hat{k}\times} \begin{pmatrix} \alpha_{11}\hat{i}\hat{i}' & \alpha_{12}\hat{i}\hat{j}' & \alpha_{13}\hat{i}\hat{k}' \\ \alpha_{21}\hat{j}\hat{i}' & \alpha_{22}\hat{j}\hat{j}' & \alpha_{23}\hat{j}\hat{k}' \\ \alpha_{31}\hat{k}\hat{i}' & \alpha_{32}\hat{k}\hat{j}' & \alpha_{33}\hat{k}\hat{k}' \end{pmatrix} e^{-\theta\hat{k}\times} = \begin{pmatrix} \alpha_{11}\hat{r}\hat{r}' & \alpha_{12}\hat{r}\hat{\theta}' & \alpha_{13}\hat{r}\hat{z}' \\ \alpha_{21}\hat{\theta}\hat{r}' & \alpha_{22}\hat{\theta}\hat{\theta}' & \alpha_{23}\hat{\theta}\hat{z}' \\ \alpha_{31}\hat{z}\hat{r}' & \alpha_{32}\hat{z}\hat{\theta}' & \alpha_{33}\hat{z}\hat{z}' \end{pmatrix} = \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times)$$

This then provides a standardized means to change base between standard rectangular and polar cylindrical (and similarly spherical), and back again, **FOR ANY BASIS FUNCTION** that is, a standardized means to change base between the standard rectangular complex number (cross-product triangle) and any derived polar system of complex number (cross-product triangle) and all of the functions which they generate, notably and most importantly, their basis exponentials, sines, cosines and tangents

$$\begin{aligned} \mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) &\rightarrow \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times) && \text{by the means of} && e^{\theta\hat{k}\times} \mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) e^{-\theta\hat{k}\times} = \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times) \\ i\times, \text{expi, cosi, sini, tani} &\rightarrow r\times, \text{exp r, cos r, sin r, tan r} \end{aligned}$$

$$j \times, \exp j, \cos j, \sin j, \tan j \rightarrow \theta \times, \exp \theta, \cos \theta, \sin \theta, \tan \theta$$

$$k \times, \exp k, \cos k, \sin k, \tan k \rightarrow z \times, \exp z, \cos z, \sin z, \tan z$$

This is a change of principal rotation axis (from i -hat, j -hat, and k -hat) to an arbitrary change in orientation (r -hat, θ -hat, and z -hat). It is still a rectangular system since the axes retain orthogonality, which is important to keep in mind: the unit basis vector of the x -axis is i -hat which is rotated **ccw** by θ and relabelled r -hat, the unit basis vector of the y -axis is j -hat which is rotated **ccw** by θ and relabelled θ -hat, and the z -axis is k -hat and is relabelled z -hat and is the axis around which the system is rotated so is identical to k -hat. For the scleronomic change of base, $\theta = \pi/2$, this is a change of principal axis (a 90 rotated basis system)

$$i \times, \exp i, \cos i, \sin i, \tan i \rightarrow j \times, \exp j, \cos j, \sin j, \tan j$$

$$j \times, \exp j, \cos j, \sin j, \tan j \rightarrow (i \times)', \exp i', \cos i', \sin i', \tan i'$$

$$k \times, \exp k, \cos k, \sin k, \tan k \rightarrow k \times, \exp k, \cos k, \sin k, \tan k$$

By the right-hand rule, the thumb is k -hat which the system rotates around by 90° , index finger pointing forward is i -hat and is rotated to left pointing and becomes j -hat, middle finger points left is j -hat and is rotated to backward pointing and becomes i -hat-conjugate (negative i -hat). Primes denote conjugates.

Everything that can

For an arbitrary scleronomic change of base ($\theta = \text{const.}$) or a rheonomic change of base ($\theta = \omega t$), all of the basis mathematics that can be done in standard rectangulars can be done similarly (including in the linear algebra sense) in polars which will be very useful for electromagnetism. I said that transforming to polar complex numbers, obtaining the basis (complex) exponentials and sines and cosines, then diadically expanding to the polar form, which was worked fully in the earlier parts of this document, and then expanding out the polar matrix bases in terms of rectangulars was hard work and then posed the rhetorical question, “is there an easier way?” and replied fortunately yes. This is it [the easier way]. This also shows that complex algebra and linear algebra are the same subjects. Linear algebra is a system of complex numbers and cross products, complex analysis is a subject of cross products and (latent) basis variables. For the linear algebra change of base formulae

$$A = PBP^{-1}$$

then with $A = \mathbf{p}(\hat{r} \times, \hat{\theta} \times, \hat{z} \times)$, $B = \mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times)$, and $P = e^{\theta \hat{k} \times}$, i.e.,

$$e^{\theta \hat{k} \times} \mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) e^{-\theta \hat{k} \times} = \mathbf{p}(\hat{r} \times, \hat{\theta} \times, \hat{z} \times)$$

The complex numbers are converted with the change of base $\hat{i} \times, \hat{j} \times, \hat{k} \times \rightarrow \hat{r} \times, \hat{\theta} \times, \hat{z} \times$

$$e^{\theta \hat{k} \times} \hat{i} \times e^{-\theta \hat{k} \times} = \hat{r} \times$$

$$e^{\theta \hat{k} \times} \hat{j} \times e^{-\theta \hat{k} \times} = \hat{\theta} \times$$

$$e^{\theta \hat{k} \times} \hat{k} \times e^{-\theta \hat{k} \times} = \hat{z} \times$$

The basis exponentials are converted with the change of base **expi, expj, expk** $\rightarrow$ **expr, expq, expz**

$$e^{\theta \hat{k} \times} \exp i(t) e^{-\theta \hat{k} \times} = \exp r(t)$$

$$e^{\theta \hat{k} \times} \exp j(t) e^{-\theta \hat{k} \times} = \exp q(t)$$

$$e^{\theta \hat{k} \times} \exp k(t) e^{-\theta \hat{k} \times} = \exp z(t)$$

The basis cosines are converted with the change of base **cosi, cosj, cosk** $\rightarrow$ **cosr, cosq, cosz**

$$e^{\theta \hat{k} \times} \cos i(t) e^{-\theta \hat{k} \times} = \cos r(t)$$

$$e^{\theta \hat{k} \times} \cos j(t) e^{-\theta \hat{k} \times} = \cos q(t)$$

$$e^{\theta \hat{k} \times} \cos k(t) e^{-\theta \hat{k} \times} = \cos z(t)$$

The basis sines are converted with the change of base **sini, sinj, sink** $\rightarrow$ **sinr, sinq, sinz**

$$e^{\theta \hat{k} \times} \sin i(t) e^{-\theta \hat{k} \times} = \sin r(t)$$

$$e^{\theta \hat{k} \times} (\sin i(t) \hat{i} \times) e^{-\theta \hat{k} \times} = \sin r(t) \hat{r} \times$$

$$e^{\theta \hat{k} \times} \sin j(t) e^{-\theta \hat{k} \times} = \sin q(t)$$

$$e^{\theta \hat{k} \times} (\sin j(t) \hat{j} \times) e^{-\theta \hat{k} \times} = \sin q(t) \hat{\theta} \times$$

$$e^{\theta \hat{k} \times} \sin k(t) e^{-\theta \hat{k} \times} = \sin z(t)$$

$$e^{\theta \hat{k} \times} (\sin k(t) \hat{k} \times) e^{-\theta \hat{k} \times} = \sin z(t) \hat{z} \times$$

The basis tangents are converted with the change of base **tani, tanj, tank** $\rightarrow$ **tanr, tanq, tanz**

$$e^{\hat{\theta}\hat{k}\times} \mathbf{tani}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanr}(t)$$

$$e^{\hat{\theta}\hat{k}\times} \left(\mathbf{tani}(t) \hat{i} \times \right) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanr}(t) \hat{r} \times$$

$$e^{\hat{\theta}\hat{k}\times} \mathbf{tanj}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanq}(t)$$

$$e^{\hat{\theta}\hat{k}\times} \left(\mathbf{tanj}(t) \hat{j} \times \right) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanq}(t) \hat{\theta} \times$$

$$e^{\hat{\theta}\hat{k}\times} \mathbf{tank}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanz}(t)$$

$$e^{\hat{\theta}\hat{k}\times} \left(\mathbf{tank}(t) \hat{k} \times \right) e^{-\hat{\theta}\hat{k}\times} = \mathbf{tanz}(t) \hat{z} \times$$

The basis secants are converted with the change of base **seci, secj, seck** → **secr, secq, secz**

$$e^{\hat{\theta}\hat{k}\times} \mathbf{seci}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{secr}(t)$$

$$e^{\hat{\theta}\hat{k}\times} \mathbf{secj}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{secq}(t)$$

$$e^{\hat{\theta}\hat{k}\times} \mathbf{seck}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{secz}(t)$$

Not surprisingly, the hyperbolic double cross basis functions transform similarly as do all basis functions of $i\times, j\times, k\times$, that is, for $\mathbf{p} = \mathbf{p}(i\times, j\times, k\times)$ for the change of base $\mathbf{p}(i\times, j\times, k\times) \rightarrow \mathbf{p}(r\times, \theta\times, z\times)$ with **expk**, that is, $e^{\hat{\theta}\hat{k}\times} \mathbf{p}(\hat{i}\times, \hat{j}\times, \hat{k}\times) e^{-\hat{\theta}\hat{k}\times} = \mathbf{p}(\hat{r}\times, \hat{\theta}\times, \hat{z}\times)$.

I do much of this on my TI (Texas Instruments 89 Titanium) and one maybe interested in the functions that I have programmed in it. For instance, each of the above can be demonstrated with the TI in the single line of working corresponding to the single line of maths such as demonstrating $e^{\hat{\theta}\hat{k}\times} \mathbf{cosi}(t) e^{-\hat{\theta}\hat{k}\times} = \mathbf{cosr}(t)$ which is the magic of the change of base formula loved and cherished in linear algebra. For the standard system, I have now 18 functions defined, starting with the basis column vectors themselves. These are

$i\text{-hat}, j\text{-hat}, k\text{-hat}, i\times, j\times, k\times$ [both singular and full rank], **expi, expj, expk, cosi, cosj, cosk, sini, sinj, sink** $i\times, j\times, k\times$

the first three being vectors, the rest matrices [since they are matrix operators that operate on either then other matrix operators or, ultimately, on a position or velocity vector (or colour vector)]. When I need a standard matrix basis component I simply calculate it from the relevant vector bases [which outer product expands it to the matrix base], for example, $i\text{-hat}$ times $i\text{-hat}$ transpose for the first diagonal matrix basis element ii' . Left hand times it by **expk** and then right hand times it by **expk** transpose and see that it equals the first diagonal polar matrix basis element. I have defined the sines (**sini, sinj, sink**) coupled with the relevant basis cross ($i\times, j\times, k\times$) to avoid the issue of the basis cross being singular. Now with the polar system, I have also programmed all of the equivalents, again with the all important unit column vectors $r\text{-hat}, \theta\text{-hat}, z\text{-hat}$. One can do this by defining/programming the basis vectors and then defining them from these, programming them directly, or by the change of base formula, the relevant function multiplied left and right by the relevant **expi, expj, or expk** and its conjugate. All of the maths in the documents I have been writing can be done easily and competently with the TI which is why I am so pleased with it [I am not into esoteric or abstract maths, the more applied it is the better]. The polar functions that I have programmed in my TI are the equivalents to the 18 standard rectangular basis functions that I had programmed previously

$r\text{-hat}, \theta\text{-hat}, z\text{-hat}, r\times, \theta\times, z\times$ [both singular and full rank], **expr expq, expz, cosr, cosq, cosz, sinr, sinq, sinz** $r\times, \theta\times, z\times$

works like a dream, especially considering the maths is otherwise very complicated in that it is involved with lots of scalar sines and cosines. I think what this means is that we will be able to do some really complicated maths without having to think too hard. What is important to bear in mind is that the polar basis functions are expressed with the polar matrix basis expanded out into scaled components of the standard rectangular basis, but are still operators for, and on, polar vector bases $r\text{-hat}, \theta\text{-hat}, z\text{-hat}$.

I will finish this document with some theorems and corollaries.

Theorem: any basis function of a common rotation axis commutes with ANY OTHER BASIS FUNCTION of the same rotation axis.

Examples

$$(1) e^{\hat{\theta}\hat{j}\times} (\mathbf{sinj} \hat{\theta} \hat{j} \times) - (\mathbf{sinj} \hat{\theta} \hat{j} \times) e^{\hat{\theta}\hat{j}\times} = 0$$

$$(2) (\mathbf{cosk} \theta) e^{\hat{\theta}\hat{k}\times} (\mathbf{sink} \theta \hat{k} \times) - (\mathbf{sink} \theta \hat{k} \times) e^{\hat{\theta}\hat{k}\times} (\mathbf{cosk} \theta) = 0$$

$$(3) (\mathbf{cosi} \theta) e^{\hat{\theta}\hat{i}\times} (\mathbf{sini} \theta \hat{i} \times) - e^{\hat{\theta}\hat{i}\times} (\mathbf{sini} \theta \hat{i} \times) (\mathbf{cosi} \theta) = 0$$

Corollary: Since these commute also with any scalar, this is why complex algebra is commutative in $i = k\times$ and so can be manipulated purely algebraically, that is, $i = k\times$ can be manipulated purely algebraically in any algebraic system of complex algebra, calculus, and analysis in general as though it was a number or algebraic scalar. This is why complex algebra can be considered an extended field of the reals despite the fact that it is not and why $i = k\times$ appears to be an algebraic scalar quantity [a number for instance] when it is definitely not a number and definitely is not a scalar quantity. It $i = k\times$ is definitely an operator that, when not explicitly, operates latently (when in algebraic systems) on unseen basis vectors and this is a consequence of the above theorem. A scalar is magnitude without direction: $k\times$ is definitely associated with direction. For example

$a\hat{i} + b\hat{j} = a\hat{i} + b\hat{k} \times \hat{i} = a\hat{i} + b(\hat{k} \times \hat{i}) = (a + (\hat{k} \times)b)\hat{i}$ which shows that whatever axis the scalar or number a is on [here $\hat{i}$ -hat but this is arbitrary and possibly latent], then b is on the mutually perpendicular axis as dictated by the cross-product triangle of the standard rectangular system or any generalized basis system with a cross-product triangle. In plane algebra, these basis variables are latent, they are there but unseen. Therefore, as per the contention in the earlier parts of this document, $i = k \times$ is definitely not a number since it is an operator and it is definitely not a scalar since it is associated with vector direction.

Corollary: the above theorem is why complex numbers [which are not numbers] commute as scalars whilst quaternions do not, that is, because basis functions do not commute unless they are of a common rotation axis. This proves that the word “number” in “complex number” is a misnomer. It is a non-commutative operator that is manipulated as though it were a number and commutes only with operators of that same principal axis, including when that axis is latent in R^2 plane algebra.

Corollary: a negative quadratic determinant maps that quantity to an orthogonal real axis in R^3 . Therefore this quantity is associated with direction so definitely does not fit the criterion for a scalar or a number.

Corollary: Since all that is required is a common rotation axis and this rotation axis can have any orientation, this is why complex numbers work, as manipulated purely algebraically, in an arbitrary coordinate system, including polar cylindrical (where the complex number $z \times$ is equivalent to that of the standard rectangular system $k \times$). The technique of cis-scalings, the resulting basis, and the basis appended with cross forming a cross-product triangle, are the quintessential features of generalized coordinate systems.

Corollary (obvious but stated nonetheless): a change of base of a basis function of a given rotation axis with respect to that same rotation axis is a null change of base. This is a consequence of the fact that the function commutes with the change of base matrices so that after a commutation the column change of base cancels the row change of base nulling the elements of the base change.

Theorem: Change of base of a product equals the product of the changes of base. The proof is easy:

$$\begin{aligned}
 e^{\theta \hat{k} \times} \left(\mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) \mathbf{q}(\hat{i} \times, \hat{j} \times, \hat{k} \times) \right) e^{-\theta \hat{k} \times} &= e^{\theta \hat{k} \times} \left(\mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) e^{-\theta \hat{k} \times} e^{\theta \hat{k} \times} \mathbf{q}(\hat{i} \times, \hat{j} \times, \hat{k} \times) \right) e^{-\theta \hat{k} \times} && \text{since } e^{-\theta \hat{k} \times} e^{\theta \hat{k} \times} = I \\
 &= \left(e^{\theta \hat{k} \times} \mathbf{p}(\hat{i} \times, \hat{j} \times, \hat{k} \times) e^{-\theta \hat{k} \times} \right) \left(e^{\theta \hat{k} \times} \mathbf{q}(\hat{i} \times, \hat{j} \times, \hat{k} \times) e^{-\theta \hat{k} \times} \right) \\
 &= \mathbf{p}(\hat{r} \times, \hat{\theta} \times, \hat{z} \times) \mathbf{q}(\hat{r} \times, \hat{\theta} \times, \hat{z} \times)
 \end{aligned}$$

which will likely prove invaluable to change of base of complicated basis functions.